quadratic forms and theta series

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Centre of Mathematical Ideas

Math. Ideas

$\underline{L} = (L, B)$ even lattice

(i.e. $L \cong \mathbb{Z}^r$, $B : L \times L \to \mathbb{Z}$ symm., bil., > 0)

$Q(x) := \frac{1}{2} B(x,x) \in \mathbb{Z}$ for $x \in L$; $B(x,y) = Q(x+y) - Q(x) - Q(y)$

Ex 1) $(\mathbb{Z}^n, Q(x+y) - Q(x) - Q(y))$ $Q \in \mathbb{Z}[x,Y]_2$ quadratic form

Associated theta series. If we view $L = \mathbb{Z}e_1 + \dots + \mathbb{Z}e_n$, then

$$\Theta_{\underline{L}} = \sum_{x \in L} q^{Q(x)} \quad (q = e^{2\pi i z},\ z \in \mathfrak{H}) \quad Q(\textstyle\sum x_i e_i) \in \mathbb{Z}[x,Y]_2$$

More generally

$$\Theta_{\underline{L}}^{(n)}(Z) = \sum_{x_1, \dots, x_n \in L} e^{\pi i \, tr((B(x_i,x_j))_{i,j} Z)} \quad (Z \in \mathfrak{H}^{(n)})$$

Θ-series of degree n

$$= \sum_{\substack{\mathbb{Q}^{n\times n} \ni T = T^t > 0 \\ half\ int.}} \underbrace{\#\{(x_1, \dots, x_n) \in L^n \mid \tfrac{1}{2}(B(x_i,x_j))_{i,j} = T\}}_{A_{\underline{L}}(T)} e^{2\pi i \, tr(TZ)}$$

counts representation numbers

Other types

Jacobi Θ-series

$$\vartheta_{\underline{L}}(z, w) = \sum_{x \in L} q^{Q(x)} e^{2\pi i B(x,w)} \quad z \in \mathfrak{H},\ w \in \mathbb{C} \otimes L$$

and similar for higher degree.

Or if $\underline{L}$ has extra structures

L $\mathbb{Z}_k$-module (k totally real number field, $n = [k:\mathbb{Q}]$)

$B(x,y) = tr_{k/\mathbb{Q}}\, b(x,y)$ ($b : L \times L \to k$ totally pos. def. scalar product)

Hilbert θ-series (of degree 1)

$$\Theta_{(L,b)}(\tau_1, \dots, \tau_n) = \sum_{x \in L} e^{\pi i \sum_{\sigma_i : k \hookrightarrow \mathbb{R}} \tau_i \sigma_i(b(x,x))} \quad (\sigma_i : k \hookrightarrow \mathbb{R})$$

Also Hilbert-Siegel, Hilbert-Jacobi (A-S-? ?),

Hermitian, Quaternionic etc.