Show that the map

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longrightarrow \sqrt[3]{x} \quad \text{is a homeomorphism .}$$

a). Show f is continuous at $x_o = 0$.

For an arbitrary $\varepsilon > 0$ choose $\delta = \varepsilon^3$. Then for each $x \in \mathbb{R}$ we have $|x-x_o| < \delta \rightarrow |x| < \varepsilon^3 \rightarrow |\sqrt[3]{x}| = |f(x)-f(x_o)| < \varepsilon$.

Hence f is continuous at $x_o = 0$.

b). Show that f is continuous at all $x_o \neq 0$.

(Hint: Use the identity $(a^2+ab+b^2)(a-b) = a^3-b^3$).

Assume $x_o \neq 0$ and $\varepsilon > 0$. For any $x \in \mathbb{R}$ which has the same sign as x_o, apply the above identity to $a = \sqrt[3]{x}$ and $b = \sqrt[3]{x_o}$ and obtain

$(x-x_o) = (\sqrt[3]{x}^2 + \sqrt[3]{x} \cdot \sqrt[3]{x_o} + \sqrt[3]{x_o}^2)(\sqrt[3]{x} - \sqrt[3]{x_o})$ and, since
$|\sqrt[3]{x}^2 + \sqrt[3]{x} \cdot \sqrt[3]{x_o} + \sqrt[3]{x_o}^2| = \sqrt[3]{x}^2 + \sqrt[3]{x}\sqrt[3]{x_o} + \sqrt[3]{x_o}^2 \geq \sqrt[3]{x_o^2} > 0$,
$|f(x) - f(x_o)| = |\sqrt[3]{x} - \sqrt[3]{x_o}| \leq |x-x_o| \cdot \dfrac{1}{\sqrt[3]{x_o^2}}$.

If we choose $\delta = \mathrm{Min}(|\frac{x_o}{2}|, \sqrt[3]{x_o^2} \cdot \varepsilon)$, then $|x-x_o| < \delta$ implies that x has the same sign as x_o, and that $|f(x) - f(x_o)| < \varepsilon$. Hence f is continuous at x_o.

c). Find a continuous inverse of f.

The map $g : \mathbb{R} \rightarrow \mathbb{R}$ with $g(y) = y^3$ is clearly an inverse of f. Moreover it is continuous since it can be obtained by threefold multiplication of the identity map with itself.

a). and b). show that f is a continuous map, by c). f has a continuous inverse, hence f is a homeomorphism.