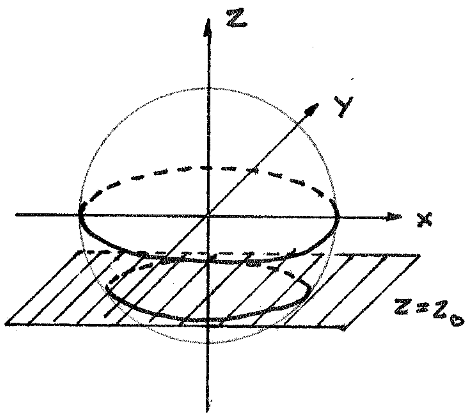
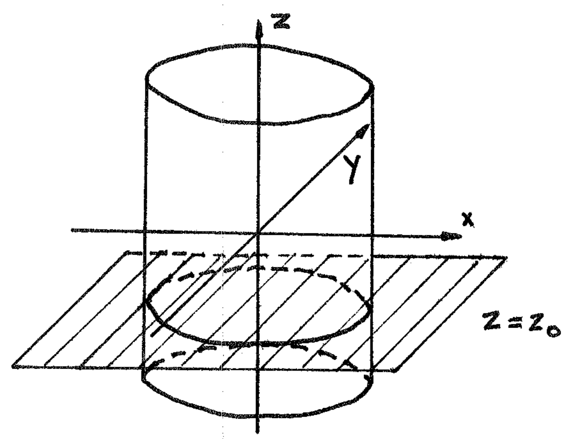
$X$ is pathwise connected, $\exists$ path $g : I \to X$ with $g(0) = x_o$ and $g(1) = x_1$. Since $f$ is continuous, $f \circ g : I \to Y$ is continuous with $f \circ g(0) = y_o$ and $f \circ g(1) = y_1$. We have shown that any two points $y_o$ and $y_1$ in $Y$ can be joined by a path, i.e. $Y$ is pathwise connected.

4). Prove that $S^2 - \{(0,0,1) \cup (0,0,-1)\}$ and $S^1 \times (-1,1)$ are homeomorphic.

Proof: $S^2 - \{(0,0,1) \cup (0,0,-1)\}$ is the sphere with top and bottom point deleted. $S^1 \times (-1,1)$ is a cylinder.




For each level $z = z_o \in (-1,1)$ the plane $z = z_o$ intersects both the sphere and the cylinder in a circle. By stretching these circles into one another, we get the following level preserving maps

$$S^2 - \{(0,0,1) \cup (0,0,-1)\} \underset{g}{\overset{f}{\rightleftarrows}} S^1 \times (-1,1)$$

$$(x, y, z) \xrightarrow{f} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$$

$$((\sqrt{1-t^2})x, (\sqrt{1-t^2})y, t) \xleftarrow{g} (x, y, t)$$

Since for $(x,y,z) \in S^2$ we have $\sqrt{x^2+y^2} = \sqrt{1-z^2}$, it is easy to see that $f \circ g = Id$ and $g \circ f = Id$. Both $f$ and $g$ are continuous since their component functions are obtained from the projection maps by such continuity preserving operations as squaring, addition, square root, divisions etc. Hence $f$ is the required homeomorphism.