2). Prove one of the following two statements:

a). The product of two Hausdorff spaces is Hausdorff.

b). The product of two pathwise connected spaces is pathwise connected. Here recall the following

<u>Definition:</u> A space X is called pathwise connected if $\forall\, x_o \in X\ \forall\, x_1 \in X\ \exists$ a continuous map ("path") $g : I \to X$ with $g(0) = x_o$ and $g(1) = x_1$.

Proof of 2a): Let X and Y be Hausdorff spaces, and (x_o,y_o), (x_1,y_1) two distinct points in $X \times Y$. Then either $x_o \neq x_1$ or $y_o \neq y_1$. In the first case $\exists$ disjoint neighborhoods U_{x_o} and U_{x_1} of x_o (resp. x_1) in X, and $U_{x_o} \times Y$ and $U_{x_1} \times Y$ are disjoint neighborhoods of (x_o,y_o) (resp. (x_1,y_1)) in $X \times Y$. A similar argument establishes the existence of disjoint neighborhoods of (x_o,y_o) and (x_1,y_1) in the case $y_o \neq y_1$. Hence $X \times Y$ is Hausdorff.

Proof of 2b): Let X and Y be two pathwise connected spaces, and let (x_o,y_o) and (x_1,y_1) be any two points in $X \times Y$. Then because of the pathwise connectedness of X and Y we have paths $g_x : I \to X$ (resp. $g_y : I \to Y$) joining x_o to x_1 (resp. y_o to y_1). The product path

$$\begin{aligned} g : I &\longrightarrow X \times Y \\ t &\longrightarrow (g_x(t),\, g_y(t)) \end{aligned}$$

is continuous (because its component maps g_x and g_y are) and joins (x_o,y_o) to (x_1,y_1). Hence $X \times Y$ is pathwise connected.

3). Let X,Y be topological spaces and $f : X \to Y$ continuous and onto. Then if X is pathwise connected, so is Y.

Proof: Let y_o and y_1 be arbitrary points in Y. Since f is onto, $\exists\, x_o,x_1 \in X$ s.t. $f(x_o) = y_o$ and $f(x_1) = y_1$. Since