$p_2 : X \times Y \to Y$.

Prove that the map

$$\begin{array}{ccc} \pi_1(X\times Y, (x_o,y_o)) & \longrightarrow & \pi_1(X,x_o) \times \pi(Y,y_o) \\ \alpha & \longrightarrow & (p_{1*}(\alpha) \quad , p_{2*}(\alpha)) \end{array}$$

is a group isomorphism (i.e. under π_1 the product of spaces corresponds to the product of groups).

π_1 allows us to associate algebraic things to topological things, and hence to translate topological problems into an algebraic setting where they are sometimes easier to solve. This is the basic idea of algebraic topology.

O. Computation of $\pi_1(S^1)$, and applications.

Let $x_o = (1,0)$ be the basepoint of the circle S^1. For $n \in \mathbb{Z}$ define $c_n : I \to S^1$ by $c_n(t) = e^{2\pi i\, nt} = (\cos 2\pi nt, \sin 2\pi nt)$ (i.e. c_n is the loop which "goes n times around the circle").

Theorem

The map

$$\begin{array}{rccc} h : & \mathbb{Z} & \longrightarrow & \pi_1(S^1, (1,0)) \\ & n & \longrightarrow & [c_n] \end{array}$$

is a group isomorphism. (Proved in class)

Corollary 1: There exists no continuous map $f : E^2 \to S^1$ such that $f(x) = x$ for all $x \in S^1$.

Proof: Assume there exists such an f. The condition on f says that $f \circ i = Id_{S^1}$, where $i : S^1 \to E^2$ is the natural inclusion. Hence $f_* \circ i_* = Id_{S^1 *} = Id_{\pi_1(S^1)}$.

$$\begin{array}{ccccc} Id = f_* i_* : \pi_1(S^1,(1,0)) & \xrightarrow{i_*} & \pi_1(E^2,(1,0)) & \xrightarrow{f_*} & \pi_1(S^1,(1,0)) \\ \wr\!\wr\!\wr & & \| & & \wr\!\wr\!\wr \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$