homotopic with fixed endpoints then so are $f \circ c$ and $f \circ \tilde{c}$. Hence the map

$$\pi_1(f) = f_* : \pi_1(X,x_o) \longrightarrow \pi_1(Y,y_o)$$
$$[c] \longrightarrow [f \circ c]$$

is well defined, and it is easy to check (from the definition of the addition in fundamental groups) that $f_* = \pi_1(f)$ is a homomorphism. Thus π_1 not only associates groups to topological spaces (with basepoints) but also π_1 associates group homomorphisms to (basepoint preserving) continuous maps.

Lemma: Let (X,x_o), (Y,y_o) and (Z,z_o) be topological spaces with basepoints, and let $f : X \to Y$ and $g : Y \to Z$ be basepoint preserving maps. Then:

a). $g_* \circ f_* = (g \circ f)_* : \pi_1(X,x_o) \xrightarrow[(g \circ f)_*]{f_* \nearrow \pi_1(Y,y_o) \xrightarrow{g_*}} \pi_1(Z,z_o)$

b). $(Id_X)_* = Id_{\pi_1(X,x_o)} : \pi_1(X,x_o) \longrightarrow \pi_1(X,x_o)$

Proof: Exercise.

It follows from this lemma that a homeomorphism $f : X \xrightarrow{\cong} Y$ induces a group isomorphism

$$f_* : \pi_1(X,x_o) \xrightarrow{\cong} \pi(Y,f(x_o))$$

In fact we have the much stronger result:

Problem 3: Let X,Y be pathwise connected, homotopy equivalent topological spaces. Then their fundamental groups are isomorphic. ∎

The construction of the fundamental group is also compatible with products:

Problem 4: Let X,Y be topological spaces with basepoints x_o and y_o, and denote the projections by $p_1 : \begin{matrix} X \times Y \to X \\ (x,y) \to x \end{matrix}$ and