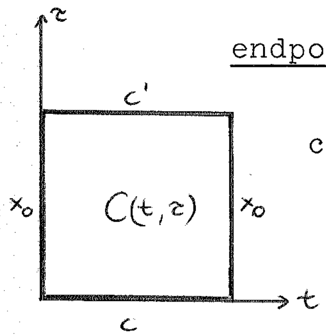
L. The fundamental group (I)

Let X be a topological space and x_o a fixed point in X referred to as base point. Then in the set of loops

$$\mathcal{L}(X,x_o) = \{c : I \to X \text{ continuous s.t. } c(0) = c(1) = x_o\}$$

there is the following equivalence relation ("homotopy with fixed endpoints"):



$$c \sim c' \iff \exists\, C : I \times I \to X \text{ continuous map}$$
$$\text{s.t. } \forall\, t \in I \quad C(t,0) = c(t) \text{ and } C(t,1) = c'(t)$$
$$\text{and } \forall\, \tau \in I \quad C(0,\tau) = C(1,\tau) = x_o \quad .$$

For a loop $c \in \mathcal{L}(X,x_o)$, we denote its equivalence class by

$$[c] = \{c' \in \mathcal{L}(X,x_o) \mid c' \sim c\} \quad \subset \mathcal{L}(X,x_o)$$

Such an equivalence class is a special subset of $\mathcal{L}(X,x_o)$, and each loop c is in exactly one equivalence class (namely in $[c]$). The set of all these equivalence classes is denoted by $\pi_1(X,x_o)$.

There is a composition of loops: for $c,\tilde{c} \in \mathcal{L}(X,x_o)$ define a new loop $c + \tilde{c} \in \mathcal{L}(X,x_o)$ by

$$c + \tilde{c} \quad : I \longrightarrow X$$

$$(c+\tilde{c})(t) = \begin{cases} c(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \tilde{c}(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

If $c \sim c'$ and $\tilde{c} \sim \tilde{c}'$, then $c + \tilde{c} \sim c' + \tilde{c}'$.

Now, let α and $\tilde{\alpha}$ be elements of $\pi_1(X,x_o)$; we can choose loops $c,\tilde{c} \in \mathcal{L}(X,x_o)$ such that $\alpha = [c]$ and $\tilde{\alpha} = [\tilde{c}]$. Then the class of $c + \tilde{c}$ does not depend on the choice of c and $\tilde{c}$, i.e. we can define without any ambiguity

$$\alpha + \tilde{\alpha} \quad = \quad [c + \tilde{c}] \quad \in \quad \pi_1(X,x_o)$$

$\pi_1(X,x_o)$, together with this addition, is called the fundamental