also have $f_o \circ g \sim f_1 \circ g : X' \to Y$ and $h \circ f_o \sim h \circ f_1 : X \to Y'$.

Exercise: Any two continuous maps from a topological space X into Euclidean space $\mathbb{R}^n$ are homotopic.

K. Homotopy equivalence

Definition: Two topological spaces X and Y are called homotopy equivalent if $\exists$ continuous maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

such that $j \circ f \sim Id_X : X \to X$ and $f \circ g \sim Id_Y : Y \to Y$.

Whereas homotopy is an equivalence relation between continuous maps (see J.), homotopy equivalence is an equivalence relation between topological spaces. We use the same notation for homotopy equivalence ("$X \sim Y$") and have:

(i) $\forall$ X topological spaces $X \sim X$

(ii) $\forall$ X, Y topological spaces $(X \sim Y \to Y \sim X)$

(iii) $\forall$ X, Y, Z topological spaces $(X \sim Y$ and $Y \sim Z \to X \sim Z)$

It is clear that homeomorphic spaces are homotopy equivalent.

Problems:

1). Establish homotopy equivalences:

$$\mathbb{R}^n \sim E^n \sim \{point\} \quad \text{for} \quad n \geq 1 .$$

2). Prove: $\mathbb{R}^n - \{0\} \sim S^{n-1}$

3). The cylinder $S^1 \times I$ is homotopy equivalent to the möbius strip.

Remark (without proof): None of the homotopy equivalent spaces in the problems 1)., 2). and 3). above are actually homeomorphic. This stresses the fact that homotopy equivalence is a much rougher relation than homeomorphy.