I. Manifolds

Definition: An n-dimensional manifold is a Hausdorff space X such that each point $x_0 \in X$ has a neighborhood in X which is homeomorphic to an open subset of $\mathbb{R}^n$. A two-dimensional manifold is called a surface.

Examples:

(i) All open subsets of $\mathbb{R}^n$ are n-dimensional manifolds.

(ii) S^n is a compact, n-dimensional manifold. (This follows easily from Problem 3). (iii) in § H.). In particular the circle S^1 is a one-dimensional manifold and the sphere S^2 is a surface.

Problems:

1). Show that the torus $T^2 \cong S^1 \times S^1$ is a surface. More generally, show that the product of two manifolds is again a manifold. What relation is there between the dimensions?

2). Show that the ellipsoid

$$\{ (x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \}$$

(where $a,b,c > 0$ are fixed real numbers) is a surface homeomorphic to S^2. Draw a sketch.

3). Show that the paraboloid

$$\{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z = 0 \}$$

is a surface. Is it homeomorphic to any of the topological spaces you are supposed to know well? Draw a sketch.

4). Introduce on S^n the following equivalence relation: for $x,y \in S^n$ $x \sim y \iff x = y$ or $x = -y$. The set P^n of equivalence classes $\{x,-x\}$ is given the quotient topology with respect to the map

$$p : S^n \to P^n$$
$$x \to \{x,-x\}$$