+1, and there exists a unique angle $\alpha$, $0 \leq \alpha \leq \pi$, such that

$$\cos \alpha = \frac{(x,y)}{\| x\| \cdot \| y\|} ,$$

We call $\alpha$ "the angle between $x$ and $y$" .

If $\alpha = \frac{\pi}{2} = 90^{\circ}$, i.e. if $(x,y) = 0$, we say $x$ and $y$ are orthogonal (or perpendicular) to one another and write $x \perp y$.

Example:

$\mathbb{R}^1 = \mathbb{R}$ = real line

$\mathbb{R}^2$ = plane described in terms of cartesian coordinates

$\mathbb{R}^3$ = three-dimensional space of cartesian coordinates .

Check that in these cases our definition of the angle between $x$ and $y$ coincides with the usual definition of the angle between the vector from the origin to $x$ and the vector from the origin to $y$. Similarly $\| x\|$ is just the usual distance of $x$ from the origin.

In $\mathbb{R}^n$ the norm $\| \ \|$ satisfies the following "norm axioms":

(i) $\forall\ x \in \mathbb{R}$ $\| x\| \geq 0$; and $\| x\| = 0 \Longleftrightarrow x = 0$

(ii) $\forall\ \lambda \in \mathbb{R}$ and $\forall\ x \in \mathbb{R}^n$ $\|\lambda x\| = |\lambda| \cdot \| x\|$

(iii) $\forall\ x \in \mathbb{R}^n$ and $\forall\ y \in \mathbb{R}^n$ $\|x+y\| \leq \| x\| + \| y\|$ (triangle inequality)

From this it is clear that the "distance function" or "metric" $\alpha$, defined by (for $x,y \in \mathbb{R}^n$)

$$d(x,y) = \|x - y\| \in \mathbb{R}$$

has the usual properties of a metric $(x,y,z \in \mathbb{R}^n)$:

(i) $d(x,y) \geq 0$; and $d(x,y) = 0 \Longleftrightarrow x = y$

(ii) $d(x,y) = d(y,x)$ ("symmetry")

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ ("triangular inequality") .

This metric $d$ defines a topology on $\mathbb{R}^n$: a subset $U$ of $\mathbb{R}^n$