* Extra Page

(0)

G acts on the space of holomorphic functions on $h^n \times \mathbb{C}^n$ via

$$f|_{k,m}\left(g(A,\omega),\lambda,\mu\right)(\tau,z) := \omega^{-2k}$$

$$f\left(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\right) e\left(-mc\frac{(z+\lambda\tau+\mu)^2}{c\tau+d} + m\lambda^2 z + 2m\lambda z\right)$$

where $e(x) = e^{2\pi i \, tr(x)}$.

Möbius transformation on the n-fold product of the upper half plane.

* Method for H/S paper:

The space $\bigoplus_{k\in\mathbb{Z}} J_{k,m}$ is a module over the ring $\bigoplus_{k\in\mathbb{Z}} M_k$ which is finitely generated as a ring for $SL(2,\mathbb{Z})$.

The space $\bigoplus_{k\in\mathbb{Z}} J_{k,m}$ is not f.g but as a module over the ring of HMF it is f.g.

work in p…
w/ Hayn… Uh

~~j…~~ ~~work in~~ pa…

View $\mathbb{C}^n$ as k-alg :

$(a,z) \mapsto (\sigma_1(a) z_1, \ldots)$

K # field tot. real

$n = [K:\mathbb{Q}]$, $\partial = \partial_K$.

①

Singular Jacobi Forms over # fields:

$$\vartheta(\tau,z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}} = q^{\frac{1}{8}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n\geq 1}(1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})$$

"Weirstrass σ-function"

→ Kac Moody algebras
~ arithmetic of elliptic curves
→ Jacobi triple product identity

Question: Do there exist "similar" functions for the JF over # fields (totally real) ?

Characterization of ϑ:

$\vartheta \in J_{\frac{1}{2},\frac{1}{2}}(\varepsilon^3)$, $\quad \varepsilon(\alpha) = \dfrac{\eta(A\tau)}{\eta(\tau)w(\tau)}$ $\quad \alpha = (A,w)$

↓ certain character of $Mp(2,\mathbb{Z})$

Notation later.

Theorem: (Skoruppa) $\vartheta(\tau,z)$ and $\dfrac{\vartheta(\tau,2z)\,\eta(\tau)}{\vartheta(\tau,z)}$ are the only JF of wt $\frac{1}{2}$ on the full modular group.

Note: These are "singular" JF's.
Definition later.

Today: We shall generalize Theorem to totally real # fields

~~For~~ that:

(1) Develope a theory of JF over tot. real # fields.
(parts in literature: O. Richter, K. Bringmann, ...)

(2) Determine all singular JF's forms for tot real # fields.

Assume $K = \mathbb{Q}$ (To minimize notations,) however I have also results for arbitrary K)

$\mathcal{O} = \mathcal{O}_K$ ($\mathcal{O}$ ring ...

$\partial = \partial_K$ diff. ..

$[K:\mathbb{Q}] = n$

$\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{R}$

Let $k \in \frac{1}{2}\mathbb{Z}$, $m \in \partial^{-1}$, $m \gg 0$

<u>Defn</u>: A JF of wt k and index m on $G^J := Mp(2,\mathcal{O}) \ltimes \mathcal{O} \times \mathcal{O}$ with character χ

($\chi : Mp(2,\mathcal{O}) \longrightarrow S^1$'s with $\ker \chi$ having finite index in $Mp(2,\mathcal{O})$)

is a hol. fn. $\phi : \mathfrak{h}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ satisfying

$$\phi|_{k,m}\, g = \chi(g)\, \phi \quad , g \in G^J.$$

$J_{k,m}(\chi) :=$ space of such ϕ.

$G := Mp(2,\mathcal{O}) = \{ (A,w) \mid A \in SL(2,\mathcal{O}),\ w : \mathfrak{h}^n \xrightarrow{\text{hol}} \mathbb{C},\ w^2(\tau) = N(c\tau+$

$$(A,w)(B,v) = (AB, w(B\tau)\, v(\tau)) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\phi|_{k,m}\alpha\,(\tau,z) = \phi\left(A\tau, \frac{z}{c\tau+d}\right) e\left(\frac{-mcz^2}{c\tau+d}\right) w^{-2k}, \qquad \alpha = (A,w)$$

$$\phi|_{k,m}(\lambda,\mu)(\tau,z) = \phi(\tau, z+\lambda\tau+\mu)\, e(+m\lambda^2\tau + 2m\lambda z)$$

$\lambda, \mu \in \mathcal{O}$, $\quad e(x) = e^{2\pi i\, \mathrm{tr}(x)}$.

Singular Jacobi Forms over # fields:

Usual JF theory

$$\vartheta(\tau, z) = \sum_{r \in \mathbb{Z}} q^{\frac{r^2}{8}} \zeta^{r/2} = q^{\frac{1}{8}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n \geq 1} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1})$$

weirstrass σ-function

- shows up in the theory of Kac-Moody Algebras
- has relations to arithmetic of elliptic curves

~~[illegible]~~ Question: Do there exist "similar functions" for the Jacobi forms over totally real number fields?

Characterization of ϑ:

$\vartheta \in J_{\frac{1}{2}, \frac{1}{2}}(\varepsilon^3)$, ε: is a certain character of ~~[illegible]~~ $(2, \mathbb{Z})$ defined by

$$\varepsilon(\alpha) = \frac{\eta(A\tau)}{\eta(\tau) w(\tau)} \quad (\alpha = (A, w))$$

Theorem: (Skoruppa) $\vartheta(\tau, z)$ and $\dfrac{\vartheta(\tau, 2z)}{\vartheta(\tau, z)} \eta(z)$ are the only JF of weight $\frac{1}{2}$ on the full modular group.

Note: These forms are "singular Jacobi forms". I will define singular forms later.

~~[illegible] gives a clue for the [illegible]~~ ~~[illegible]~~ Main Problem: Such a thm can be gen to # fields?

This theorem gives a clue for the Main problem.

~~[illegible]~~ (1) Developping a theory of JF for a given totally real # field K (partly results!)

(2)

ind. cop. of $W(\underline{M})$

$$\leq \dim \mathrm{Hom}_G\big(W(\underline{M}), W(\underline{M})\big)$$

$$\simeq \big(W(\underline{M})^* \otimes W(\underline{M})\big)^G$$

$$\simeq \big(W(\underline{M}(-1)) \otimes W(\underline{M})\big)^G$$

$$\simeq W(\underline{M}(-1) \oplus \underline{M})^G$$

$\simeq$ set of isotropic subgroups of $\underline{M}(-1) \oplus \underline{M}$ — Nebe & Sloane, Rains

if $\underline{M} \simeq \left(\mathbb{Z}/2m\mathbb{Z}, \frac{x^2}{4m}\right)$

$$\geq \sigma_0(m\partial)$$

a cop.

irr. cop. of $W(\underline{M})$

$$\leq \dim \mathrm{Hom}_G\big(W(\underline{M}), W(\underline{M})\big)$$

$$= \sigma_0(m\partial)$$

Lemma

(In all these formulas)

$\mathbb{C}^n$ is considered as K-algebra via

$$(a, z) \longmapsto (\sigma_1(a) z_1, \ldots \sigma_n(a) z_n).$$

So: $A\tau$, $\frac{z}{c\tau+d}$, ... are meaningful.

Moreover, we put

$$tr(az) := \sum_{i=1}^{n} \sigma_i(a) z_i \qquad (a \in \mathcal{O})$$

$$N(az) := \prod_{i=1}^{n} \sigma_i(a) z_i$$

Examples of JF hard to find.

~~~ / Skoruppa / Hayashida~~ ; ~~Explicit~~ construction of examples of JF's.

Let $\phi \in J_{k,m}^{k}(\chi)$.

Easy to see that ϕ is periodic wrt τ and z. ϕ has a ~~Because of this we have~~ Fourier Expansion

~~Df.~~ namely

$$\phi(\tau, z) = \sum_{\substack{r \in \mathfrak{d}^{-1} \\ t \in U^{\#}}} c_\phi(t, r)\, q^t \zeta^r, \text{ where}$$

$q^t(\tau) = e(t\tau)$, $\zeta^r(z) = e(rz)$,

$U = \{ b \in \mathcal{O} \mid \chi(T^b) = 1$ and $U^{\#}$ is the dual
$T^b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

of U wrt the trace.
~~~

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$U \underset{f.i}{\subset} \Theta$ since $\Theta/U \simeq Mp(\mathbb{Z}\Theta)/\ker \chi$.

(Koecher Principle)

For $k \neq \mathbb{Q}$: $c_\phi(t,r) = 0$ unless $4mt - r^2 = 0$ or $4mt - r^2 >> 0$ (easy cons. of $\frac{1}{2}$ ϑ-exp)

(For $k = \mathbb{Q}$ this condition should be added to (*)

Defn: ϕ is called singular if $c_\phi(t,r) \neq 0$ only if $4mt - r^2 = 0$.

Lemma: ~~[illegible]~~

$\phi \neq 0$ is singular ~~iff~~ ϕ has wt $\frac{1}{2}$.

Pf: Easy consequence of theta expansion. $\square$

Theorem: ϕ has theta expansion of the form

$$\phi(\tau,z) = \sum_{x \in \partial^{-1}/2m\mathcal{O}} h_x(\tau)\, \vartheta_{m,x}, \text{ where}$$

h_x are HMF of ~~[illegible]~~ $k - \frac{1}{2}$, $\vartheta_{m,x}(\tau,z) = \sum_{\substack{r \in \partial^{-1} \\ r \equiv x \bmod 2m}} q^{\frac{r^2}{4m}} \zeta^r$

Pf: Follows from the ~~[illegible]~~ transformation law wrt ~~[illegible]~~ $\mathcal{O} \times \mathcal{O}$. $\square$

Note: 1) $\vartheta_{m,x}$ are JF of wt $\frac{1}{2}$ on some ~~[illegible]~~ subgroup. (~~[illegible] Poisson summation formula~~

2) $Th_m := \mathrm{span}\langle \vartheta_{m,x} \mid x \in \partial^{-1}/2m\mathcal{O} \rangle$ is a G-module via $\frac{1}{2}, m$ - action;

Lemma) 1-dim. G-subbundles of Th_m correspond to singular JF on G.
Pf: Note exp. $\square$

(1 dim pres $\leftarrow$ decom into irr. $\leftarrow$ weil repr.)

$$\underline{M}(m) := \left(\partial^{-1}/_{2m\mathcal{O}}, \frac{-N(x)^2}{4m} \right)$$

f.q. dom.: $\underline{M} = (M, Q)$, M ~~finite~~ $\mathcal{O}$-module
$Q : M \to K/\partial^{-1}$ q.f. M.

because, as left G-modules

Theorem: $Th_m^V \xrightarrow{\cong} W(\underline{M}(m))$. $\uparrow$ weil rep. ass. $\underline{M}(m)$.
$(\mathbb{C}(1)^2 ; (\alpha, \vartheta) \mapsto \alpha \cdot \vartheta := \vartheta|_{\alpha} \, \partial^{-1})$

(More explicitly),
$W(\underline{M}(m))$: left G-module $V := \mathbb{C}[\partial^{-1}/_{2m\mathcal{O}}]$

with the G-action given by:

$$(g, e_x) \longmapsto \rho_{\underline{M}(m)}(e_x).$$

Here
$$\rho_{\underline{M}(m)}(T^b, 1)(e_x) = e\left(\frac{-b x^2}{4m}\right) e_x$$

$$\rho_{\underline{M}(m)}(S, \sqrt{\tau})(e_x) = \frac{1}{\sqrt{|\partial^{-1}/_{2m\mathcal{O}}|}} \, \sigma(\underline{M}(m)) \sum_{y \in \partial^{-1}/_{2m\mathcal{O}}} e\left(\frac{xy}{2m}\right) e_y$$

(~~$B(x,y) := Q(x+y) - Q(x) - Q(y)$~~)

where
$$\sigma(\underline{M}(m)) = \frac{1}{\sqrt{|\partial^{-1}/_{2m\mathcal{O}}|}} \sum_{x \in \partial^{-1}/_{2m\mathcal{O}}} e\left(\frac{x^2}{4m}\right)$$

(weil ~~reps / class by N. sts~~)

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Lemma: ~~Note that~~ G is generated by $(T^6, 1)$ and $(S, \sqrt{N}Z)$ (~~Vasserstein~~)

Pf: (Vaserstein)
(N)

Theorem: (1) $W(\underline{M}(m)) \cong \bigoplus_{(d)^2 | m\partial} \bigoplus_{(f) | \frac{m\partial}{d^2}, \ f \ \square\text{-free}} W(\underline{M}(m/d^2)_f)^{\mathrm{new}}$

as G-modules.

(2) For each m, f, $W(\underline{M}(m_f))^{\mathrm{new}, f}$ is irr.

New part of $W(\underline{M}(m))$ is the orthogonal ~~complement~~ of

$$\sum_{\substack{(d)^2 | m\partial \\ (d) \neq 0}} i_d \, W(\underline{M}(\tfrac{m}{d^2}))$$

wrt the scalar product!

$\langle e_x | e_y \rangle = \delta_{xy}$.

$i_d(e_x) = \sum_{\substack{y \in \mathcal{O}/2m\mathcal{O} \\ y \equiv x \bmod \frac{2m}{d^2}\mathcal{O}}} e_y$.

(1)

(Idea of the Pf: $W(\underline{M}(m))$: sum of Weil rep. of quotients of fg. $\mathcal{O}$-mod, our case quotients corr. to summands, and then these quotients can be furth decom. using the action of f.o.s.)

$O(m) := \{ \varepsilon \bmod 2m\partial \mid \varepsilon^2 \equiv 1 \ 4m\partial \}$

on $W(\underline{M}(m))$ via

$(\varepsilon, e_x) \mapsto e_{\varepsilon x}$. This action commutes with the G-action.

So, $\omega(\underline{M}(m))^{f_1} = \{ v \in \omega(\underline{M}(m)) \mid \varepsilon v = \chi_f(\varepsilon) v, \ \forall \varepsilon \in \mathcal{O}_m \}$

~~is~~ invariant under G.

Here $\chi_f(\varepsilon) = \mu\left(\left(f, \frac{\varepsilon+1}{2}\right)\right)$.

In fact, we have $\widehat{\mathcal{O}_m} = \{ \chi_f \mid (f) / m\partial \}$, f - D free.

Idea for (2):

- Estimate for the # irr. G-modules: (# of irr. components $\leq \mathrm{Hom}_G(\omega(\underline{M}), \omega(\underline{M}(m))) = \sigma_0(m\partial)$ Theorem)
- Counting # of summands
- e.s. summands are nonzero (as follows from calculation or explicit dimension formula ← Theorem (~))

(We use dim. formula to determine 1-dim. G-subbundles of $\omega(\underline{M}(m))$ which are in 1-1 correspondence with 1-dim. G-sub. of $\mathrm{Th}^\vee$ via.

Lemma:

$$\begin{array}{ccccc}
\omega(\underline{M}(m)) & \xrightarrow{\ \simeq\ } & \mathrm{Th}_m^\vee & & \vartheta(\tau, \cdot) \\
\uparrow & & \uparrow U_d & & \downarrow \\
\omega\left(\underline{M}\left(\frac{m}{d^2}\right)\right) & \xrightarrow{\ \simeq\ } & \mathrm{Th}_{\frac{m}{d^2}}^\vee & & \vartheta(\tau, \cdot)
\end{array}$$

next $\longrightarrow$

(7)'

~~Hence we get~~

Theorem; $T_{hm} = \bigoplus_{(d^2)/md} \bigoplus_{f/\frac{md}{d^2}} U_d T_{h\frac{m}{d^2}}^{new, f}$

Note: ~~that~~ the summands do not depend on ~~the generator of (d).~~

Here T_{hm}^{new} is the OC of

$$\sum_{\substack{(d)^2/md \\ (d)\neq \theta}} U_d T_{h\frac{m}{d^2}}^{new, f} \quad \text{w.r.t scaler product}$$

$$\langle \psi_{m,x} | \psi_{m,y} \rangle = \begin{cases} 0 & x\neq y \\ 1 & x=y \end{cases}.$$

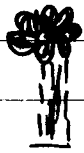
Definition; A character ideal is an ideal of the form $q_1^3 \cdots q_s^3 p_1 \cdots p_t$ where q_i's are prime ideals over 3 of degree 1 and q_i's are prime ideals ~~over~~ 2 of degree 1 with ramification index 1.

Theorem; (WWT) T_{hm} contains 1 dimensional G-submodules iff $\exists$ deg s.t. $(d)^2 / md$ s.t. $\frac{4md}{d^2}$ is a character ideal.

Pf: uses the 1-1 correspondence of 1-dim spaces of $\mathcal{U}(M(m))$ and T_{hm} and the criterion for the 1-dim G-sub. of $\mathcal{U}(M(m))$ that we obtained using the dim formula.

__Theorem (?)__ If $4m\partial$ is a character ideal and $f = p_1 \cdots p_s\, q_1^3 \cdots q_t^3$ (as before) then the space $\mathrm{Thm}^{\mathrm{new}, f} = \mathbb{C}\, \vartheta_c$ where

$$\vartheta_c = \sum_{s \in \mathcal{O}} \chi_c(s)\, q^{\frac{r^2}{4mf^2}}\, \zeta^{s/f} \quad \text{where}$$



$\partial = (f)$.

Here $\chi_c = \chi_{p_1} \cdots \chi_{p_s} \chi_{q_1^2} \cdots \chi_{q_t^2}$, where χ_{p_i} is the unique nontrivial character of $(\mathcal{O}/p_i)^* \simeq \mathbb{F}_3^* \simeq \mathbb{Z}/2\mathbb{Z}$ and $\chi_{q_j^2}$ is the non-trivial character of $(\mathcal{O}/q_j^2)^* \simeq \mathbb{Z}/2\mathbb{Z}$.

In fact χ_c can be extended to all $\mathcal{O}$ by setting $\chi_c(a) = 0$ for $\gcd(a, c) \neq 1$.

__Open problem__ Does ϑ_c have a Π-expansion? (work in progress).

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"Weirstrass σ-function"

→ Kac Moody algebras (Kac-Weyl denominator formula)
~ arithmetic of elliptic curves ($\vartheta(\tau,\cdot)$ is the Green's function for $\mathbb{C}/\mathbb{Z}\tau+\mathbb{Z}$)
→ Jacobi triple product identity

Question: Do there exist "similar" functions ~~for the~~ in the theory of JF over # fields (totally real) ?

Characterization of ϑ:

$\vartheta \in J_{\frac{1}{2},\frac{1}{2}}(\varepsilon^3)$, $\quad \varepsilon(\alpha) = \dfrac{\eta(A\tau)}{\eta(\tau)w(\tau)}$ $\quad \alpha = (A,w)$

↓
certain character of $Mp(2,\mathbb{Z})$

Notation later.

Theorem: (Skoruppa) $\vartheta(\tau,z)$ and $\vartheta^*(\tau,z) = \dfrac{\vartheta(\tau,2z)\,\eta(\tau)}{\vartheta(\tau,z)}$ are the only JF of wt $\frac{1}{2}$ on the full modular group.

Note: These are "singular" JF's.
Definition later.

Goal: ~~Idea~~: We shall generalize Theorem to totally real # fields

2

For that:

(1) Develope a theory of JF over tot. real # fields, K.
(parts in literature: ~~sta~~ O. Richter, K. Bringmann, ...)

(2) Determine all singular JF's forms for tot real # fields.

Assume $K = \mathbb{Q}$ (To minimize notations,) however I have also results for arbitrary K)

$\mathcal{O} = \mathcal{O}_K$ (O ring ..
$\partial = \partial_K$ diff ..
$[K:\mathbb{Q}] = n$, $\sigma_1, \dots, \sigma_n$; $k \ll m$
Let $k \in \frac{1}{2}\mathbb{Z}$, $m \in \partial^{-1}$, $m \gg 0$

<u>Defn</u>: A JF of wt k and index m on G^J $\mathrm{Mp}(2,\mathcal{O}) \ltimes \mathcal{O}\times\mathcal{O}$ with character χ
($\chi : \mathrm{Mp}(2,\mathcal{O}) \longrightarrow S^1$ with $\ker\chi$ having finite index in $\mathrm{Mp}(2,\mathcal{O})$)
is a hol. fn. $\phi : \mathfrak{h}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$ satisfying

$$\phi|_{k,m}\, g = \chi(g)\,\phi \quad , g \in G^J.$$

$J^k_{k,m}(\chi) :=$ space of such ϕ.

$G := \mathrm{Mp}(2,\mathcal{O}) = \{ (A,w) \mid A \in SL(2,\mathcal{O}),\ w : \mathfrak{h}^n \xrightarrow{hol} \mathbb{C},\ w^2(\tau) = N(c\tau+d) \}$

$$(A,w)(B,v) = (AB, w(B\tau)\,v(\tau)) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\phi|_{k,m}\alpha\,(\tau,z) = \phi\left(A\tau, \frac{z}{c\tau+d}\right) e\left(\frac{-mcz^2}{c\tau+d}\right) w^{-2k}, \qquad \alpha = (A,w)$$

$$\phi|_{k,m}(\lambda,\mu)(\tau,z) = \phi(\tau, z+\lambda\tau+\mu)\, e(+m\lambda^2\tau + 2m\lambda z)$$

$\lambda, \mu \in \mathcal{O}$, $\quad e(x) = e^{2\pi i\, \mathrm{tr}(x)}$.

(~In all these formlas)

$\mathbb{C}^n$ is considered as K-algebra via

$$(a, z) \longmapsto (\sigma_1(a) z_1, \cdots \sigma_n(a) z_n).$$

So! Az, $\frac{z}{cz+d}$, … are meaningful.

Morever, we put

$$tr(az) := \sum_{i=1}^{n} \sigma_i(a) z_i \qquad (a \in \mathcal{O})$$

$$N(az) := \prod_{i=1}^{n} \sigma_i(a) z_i$$

Examples of JF hard to find.

~ / Skoruppa / Hayashida ; ~~Explicit~~ explicit construction of examples of JF's. We determine the structure of the space $\bigoplus_{k \in \mathbb{Z}} J_{k,m}$ ($m \in \mathfrak{d}^{-1}$) as a module over $\bigoplus_{k \in \mathbb{Z}} M_{k,m}$. It is f.g. with 8 generators.

Let $\phi \in J_{k,m}^{K}(\chi)$.

Easy to see that ϕ is periodic wrt τ and z. ϕ has a ~~Because of this it have~~ Fourier Expansion

~~So~~ Namely

$$\phi(\tau, z) = \sum_{\substack{r \in \mathfrak{d}^{-1} \\ t \in \mathcal{U}^{\#} h + \mathfrak{d}^{-1}}} c_\phi(t, r)\, q^t \zeta^r, \text{ where}$$

where $h \in K$ satisfying $\chi(T^b) = e(hb)$

$q^t(\tau) = e(t\tau)$, $\zeta^r(z) = e(rz)$,

~~$\mathcal{U} = \{ b \in \mathcal{O} \mid \chi(T^b) = 1$ and $\mathcal{U}^{\#}$ is the dual of $\mathcal{U}$ wrt the trace.~~ ~~$T^b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$~~

~~$U \subset \Theta$ since $\Theta/U \cong M_p(4\Theta)/\ker X$~~
f.i.

(Koecher Principle)
(easy cons. of ϑ-exp.)

For $k \neq \Theta$: $c_\phi(t,r) = 0$ unless $4mt - r^2 = 0$ or $4mt - r^2 > 0$.

(For $k = \Theta$ this condition should be added to $\circledast$)

<u>Defn:</u> ϕ is called singular if $c_\phi(t,r) = 0$ ~~only if~~ unless $4mt - r^2 = 0$.

<u>Lemma:</u> ~~$\phi \neq 0$ has ...~~

$\phi \neq 0$ is singular ~~iff~~ ϕ has wt $\frac{1}{2}$.

<u>Pf:</u> Easy consequence of theta expansion. $\square$

<u>Theorem;</u> (ϑ-expansion) ϕ has ~~theta~~ an expansion of the form

$$\phi(\tau,z) = \sum_{x \in \mathfrak{d}^{-1}/2m\Theta} h_x(\tau)\, \vartheta_{m,x}, \text{ where}$$

h_x are HMF of ~~wt~~ $k - \frac{1}{2}$, $\vartheta_{m,x}(\tau,z) = \sum_{\substack{r \in \mathfrak{d}^{-1} \\ r \equiv x \bmod 2m\Theta}} q^{\frac{r^2}{4m}} \zeta^r$

~~<u>Pf:</u> Follows from the ~~ transformation ~~form wrt $\Theta \times \Theta$. $\square$~~

<u>Note:</u> 1) $\vartheta_{m,x}$ are JF of wt $\frac{1}{2}$ (on some ~~...~~ subgroup. ~~(Poisson summation formula~~

2) $Th_m := \text{span}\langle \vartheta_{m,x} \mid x \in \mathfrak{d}^{-1}/2m\Theta \rangle$ is a G-module via $\frac{1}{2},m$ - action,

<u>Lemma</u>: 1-dim. G-subbundles of $\mathcal{Th}_m$ correspond to singular JF on G.

Pf: Note exp. D

(1 dim piece ⟸ decom into irr. ⟸ weil rep.)

$\underline{M}(m) := \left(\partial^{-1}/_{2m\mathcal{O}}, \frac{-N(x)^2}{4m} \right)$

f.g. $\mathcal{O}$-m. : $\underline{M} = (M, Q)$, M ~~finite~~ $\mathcal{O}$-module

$Q : M \rightarrow K/\partial^{-1}$ qf/M

because, ≅ as left G-modules

<u>Theorem</u>: $\mathcal{Th}_{m,x}^{V} \xrightarrow{\cong} W(\underline{M}(m))$.

$(\mathbb{C}^2 : (\alpha, \vartheta) \mapsto \alpha \cdot \vartheta := \vartheta|_{\alpha} \, 2^{-1})$ ↑ weil rep. ass. $\underline{M}(m)$.

(More explicitely),

$W(\underline{M}(m))$; left G-module $V := \mathbb{C}[\partial^{-1}/_{2m\mathcal{O}}]$

with the G-action given by:

$$(g, e_x) \longmapsto \rho_{\underline{M}(m)}(e_x).$$

Here $\rho_{\underline{M}(m)}(T^b, 1)(e_x) = e\left(\frac{-b x^2}{4m}\right) e_x$

$\rho_{\underline{M}(m)}(S, \sqrt{z})(e_x) = \frac{1}{\sqrt{|\partial^{-1}/_{2m\mathcal{O}}|}} \sigma(\underline{M}(m)) \sum_{y \in \partial^{-1}/_{2m\mathcal{O}}} e\left(\frac{\cdots}{2m}\right)$

(~~$B(x,y) := Q(x+y) - Q(x) - Q(y)$~~)

where $\sigma(\underline{M}(m)) = \frac{1}{\sqrt{|\partial^{-1}/_{2m\mathcal{O}}|}} \sum_{x \in \partial^{-1}/_{2m\mathcal{O}}} e\left(\frac{x^2}{4m}\right)$

(weil ~~reps. / $\mathbb{Q}$ class. $\leftrightarrow$ N. sh... F...~~)

We have a theory of W(fin. O-modules); in particular, we studied their decomposition into irred. submodules. Translating these results back to Th_m gives the following:

<u>Thm</u>

$$Th_m = \bigoplus_{(d^2)\mid mD} \; \bigoplus_{f \mid \frac{mD}{d^2}} U_d \, Th^{new,f}_{\frac{m}{d^2}},$$

where $Th^{new,f}_{m} \subseteq Th_m$ is a certain $\overset{\text{irreducible}}{\vee}$ $Mp(\mathbb{Z}_0)$-submodule and U_d is the $Mp(\mathbb{Z}_0)$-~~homomorphism~~ linear map defined by $U_d : \vartheta(\tau, z) \to \vartheta(\tau, dz)$.

<u>Theorem</u> (C) If $4m\mathfrak{d}^{\mathbb{Z}:\mathfrak{C}/2}$ is a character ideal and $f = p_1 \cdots p_s\, q_1^{*} \cdots q_t^{*}$ (as before) then the space $Th_m^{\mathrm{rev}, f} = \mathbb{C}\,\vartheta_\tau$ where

$$\vartheta_\tau = \sum_{s \in \mathcal{O}} \chi_\tau(s)\, q^{\frac{r^2}{4mf^2}}\, \zeta^{s/f}$$

where $\mathfrak{d} = (f)$.

Here $\chi_\tau = \chi_{p_1} \cdots \chi_{p_s} \chi_{q_1^2} \cdots \chi_{q_t^2}$, where χ_{p_i} is the ~~unique~~ nontrivial character of $(\mathcal{O}/p_i)^* \simeq \mathbb{F}_3^* \simeq \mathbb{Z}/2\mathbb{Z}$ and $\chi_{q_j^2}$ is the non-trivial character of $(\mathcal{O}/q_j^2)^* \simeq \mathbb{Z}/2\mathbb{Z}$.

In fact χ_τ can be extended to all $\mathcal{O}$ by setting $\chi_\tau(a) = 0$ for $\gcd(a, \tau) \neq 1$.

<u>Open problem</u> Does ϑ_c have a Π-expansion? (work in progress).

One can combine (A) – (C) to a statement like $J_{\frac{1}{2}, m}(X) = 0$ unless m, X ...

I skip this statement because it would need more notation.

If ~~rk~~ $\mathrm{rk} > 1$, similar results hold. But we need lattice index JFs.

Definition: ($\mathcal{O}$-lattice)

An $\mathcal{O}$-lattice is a pair $\underline{L}=(L,\beta)$ where L is a f.g. torsion-free $\mathcal{O}$-module and β is a symmetric, non-degenerate bilinear form on $L \times L$ taking values in $\mathfrak{d}^{-1}$ s.t. $\sigma_i(\beta(x,x)) > 0$ $\forall i = 1, \dots, n$.

Definition: (Jacobi forms over K)

Let $k \in \frac{1}{2}\mathbb{Z}$, χ be a character of $Mp(2,\mathcal{O})$, $\underline{L}=(L,\beta)$ be an $\mathcal{O}$-lattice.

The space of Jacobi forms $J_{k,\underline{L}}(\mathcal{O},\chi)$ of wt k, index $\underline{L}$, character χ consists of all holomorphic functions $\phi(\tau,z)$ ($\tau \in \mathbb{H}^n$, $z \in \mathbb{C} \otimes_{\mathcal{O}} L$) which satisfy the following properties: $\mathbb{H} = \{\tau \in \mathbb{C}\otimes K \mid \mathrm{Im}(\sigma_i(\tau)) > 0 \ \forall \sigma_i : K \to \mathbb{R}\}$, $\mathbb{C} \otimes_{\mathbb{Q}} K$

For $\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, w\right)$ in $Mp(2,\mathcal{O})$ one has

- $$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) e\left(\frac{-c\beta(z)}{c\tau+d}\right) w(\tau)^{-2k} = \chi(A,w)\,\phi(\tau,z)$$

For all $x, y \in L$ one has

$$\phi(\tau, z + x\tau + y)\, e(\tau\beta(x) + \beta(x,z)) = e(\beta(x+y))\,\phi(\tau,z).$$

[~~For $K=\mathbb{Q}$, we impose the additional condition that ϕ is hol. at ∞. ($\forall\, n, r$: $c(n,r) \neq 0 \Rightarrow n \geq \beta(r)$)~~]

- For $K = \mathbb{Q}$ we impose the additional condition that ϕ is hol. at ∞.

Why do we study lattice index JFs:

In general, the map

$$\mathfrak{d}^{-1}_{\gg 0} \to \{\text{rank } 1 \text{ lattices} / \text{isometry}\}$$
$$m \longmapsto [(\mathcal{O}, m\,xy)] \quad \text{is not } \textbf{surjective}.$$

It means there are rank 1 Jacobi forms of ~~index~~ rank 1 which can not be expressed by a totally positive element in $\mathfrak{d}^{-1}$. We need these when we **decompose** the space of theta functions.

Some old forms come from lattice index JFs.

Also, we believe that there is a lifting from JF over number fields into HMF. In a current project (with Skoruppa) we generate explicit examples of such liftings. If we would not have lattice index we there would be HMF which do not correspond to any Jacobi forms.

Optional: We try to develop a systematic theory of JF/K.

- There is also current work of Strömberg, Skoruppa: they calculate the dimension of the space of Jacobi forms over K.
- Hecke theory (to do)
- Liftings of JFs $\longrightarrow$ HMF
- $K = \mathbb{Q}(\sqrt{5})$ (with Hayashi, Skoruppa) (for me?) we determine the structure of $\oplus_{k \in \mathbb{Z}} J_{k,m}$ as a module over the ring $\oplus_{k \in \mathbb{Z}} M_k$ of HMF.

~~Jacobi Forms over number fields~~

~~Singular Jaco~~

~~The function $f(\tau, z)$~~.

Current work with Skoruppa:

~~Studying the lattice~~

We try to develop a systematic theory of lattice index Jacobi forms over number fields:

- Dimension formula (~~...~~ Stromberg, Skoruppa)
  ~~generator~~
  (uses Eichler–Selberg trace formula)
- Hecke Theory (to do)
- ~~The~~ Lifting of rank 1 lattice index ~~JF~~ to HMF (current work ~~computing~~ search explicit examples)

we have a method

exp! JF (Jw ~~H~~ayashida $\mathbb{Q}(\sqrt{5})$)

If class number > 1, similar results hold. But we need lattice index JFs.

Defn: ($\mathcal{O}$-lattice)

~~Defn~~: (Lattice index JF)

~~Why lattice index?~~

For class number $h = 1$, no need lattice index.

~~Namely, let $\underline{L} = (L, \beta)$ be a lattice of rank 1.~~

~~If $h = 1$, then $\underline{L} = \mathcal{O}a$. Then $\beta(\alpha, a) = \beta(a,a)$~~

β $J_{k, \underline{L}}(\alpha) = J_{k, \frac{1}{2}\beta(a,a)}(\alpha)$

$(x \in \mathbb{C}$, $x = x'a$

$\underline{k=Q}$ $(\cancel{?}-2)$

$$J_{k,m} \xrightarrow[\text{equiv.}]{\text{Hecke}} M_{2k-2}(m)$$

$$\left\{ J_{k,m} \ni \phi \begin{array}{c}\text{Hef.}\\ \text{new. f.}\end{array} \right\} \underset{\text{can. bij.}}{\overset{\sim}{\longleftrightarrow}} \left\{ f \in M^{-}_{2k-2}(m) \right\}$$

up t. C^* Hef. new. f. up t. C^*

$\underline{\underline{k=2}}$

$$\phi \longleftrightarrow f = \sum a(n) q^n$$

$$L\left(f \otimes \left(\tfrac{D}{\cdot}\right), s\right)$$ D fund. dis. <0

$$= \sum \frac{a(n)\left(\frac{D}{n}\right)}{n^s}$$

$$\sum a(n)\left(\tfrac{D}{n}\right) q^n$$

$$\frac{\left|C_\phi(D,r)\right|^2}{|\phi|^2} = \text{cst.} \frac{L\left(f \otimes \left(\frac{D}{\cdot}\right), 1\right)}{|f|^2}$$

$$\forall\, r \equiv D \;(4m)$$

$$E/\mathbb{Q} \rightsquigarrow f \in \cancel{S}_2(m)$$

cond. m new. Hef.

$$E_D \rightsquigarrow f \otimes \left(\tfrac{D}{\cdot}\right)$$

$$\begin{pmatrix} E: y^2 = f(x) \\ E_D: Dy^2 = f(x) \end{pmatrix} \qquad L(E_D, s) = L\left(f \otimes \left(\tfrac{D}{\cdot}\right), s\right)$$

$$L(E_D, 1) \underset{\text{B-S-conj.}}{=} 0 \text{ iff } \#E_D(\mathbb{Q}) = \infty$$

— Jacobi Forms over Number Fields —

- Motivation
  - Correspondence to HMF
- Historical Sketch (Olav Richter, Kathrin Bringmann, Shuichi)
- Need of introduction of lattice index
- Careful Definition of lattice index Jacobi forms
- ~~Relation~~ JF as VVHMF (Theta-expansion principle)
- Dimension Formula (consider pure weight)

$$\left(\dim S_k(V) = \text{expl. formula for } k > 2 \right) \Rightarrow$$

$$\dim J_{k,\underline{L}}(x) \;\hat{=}\; S_{k-\frac{r}{2}}(V) \text{ for } k - \frac{r}{2} > 2 \quad (\text{i.e. } k > \tfrac{r}{2} + 2)$$

$$= \text{expl. formula}$$

theta-exp. $\Rightarrow$ $S_{k,\underline{L}}(x) = 0$ if $k - \frac{r}{2} \leq 0$ (i.e. $k < \frac{r}{2}$)

rem. cases:

$k = \frac{r}{2}$ singular wt. !

$k = \frac{r}{2} + \frac{1}{2}$ critical wt.

$k = \frac{r}{2} + 1$

$k = \frac{r}{2} + \frac{3}{2}$

$k = \frac{r}{2} + 2$

- Explicit theory of singular JF (no proofs, just expl.)