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~~Hecke Gauss Sums & Quadratic Reciprocity~~:

~ Recent Results on Hecke Gauss Sums ~

§0. Basics:

Hecke, in his book "The Theory of Algebraic Numbers" defined the sums, socalled, Hecke Gauss Sums, for a number field K and a nonzero element $\omega \in K$ in the following way:

$$C(\omega) = \sum_{\mu \bmod \mathfrak{a}} e(tr(\mu^2\omega)).$$

Here, we have $e(\cdot) = \exp(2\pi i\,\cdot)$ and $\mathfrak{a}$ denotes the denominator of $\omega\partial$, where ∂ denotes the different of K.

<u>Problem:</u> Explicit formula for $C(\omega)$ (of course known for $K = \mathbb{Q} \rightsquigarrow$ due to Gauss).

Easy to calculate the absolute value and decide when the sum vanishes. Next result shows this.

<u>Lemma:</u> The followings are equivalent:

(i) $C(\omega) \neq 0$

(ii) The group homomorphism $\tilde{\mathfrak{a}} = \frac{\mathfrak{a}}{(\mathfrak{a},2)} \longrightarrow \{\pm 1\}$, $\mu \longmapsto e(tr(\omega\mu^2))$ is trivial.

If $C(\omega) \neq 0$ then $e(tr(\mu^2\omega))$ depends only on $\mu \bmod \tilde{\mathfrak{a}}$, and $|C(\omega)| = \sqrt{N\tilde{\mathfrak{a}}} \cdot \frac{N\mathfrak{a}}{N\tilde{\mathfrak{a}}}$

- In the rest, we assume $C(\omega) \neq 0$, i.e. $tr(\mu^2\omega)$ is integral for all $\mu \in \tilde{\mathfrak{a}}$.

We set

$$B(\omega) := \frac{1}{\sqrt{N\tilde{\mathfrak{a}}}} \sum_{\mu \bmod \tilde{\mathfrak{a}}} e(tr(\mu^2\omega)).$$ Hence, from Lemma $|B(\omega)| = 1$.

~~In~~ fact, $B(\omega)$ is an eighth root of unity. (2)

~~Namely~~ It is easy to show that

$\underline{M}_\omega := (\mathcal{O}/\tilde{a}, \ \mu + \tilde{a} \longmapsto tr(\omega\mu^2) + \mathbb{Z})$ is a non-degenerate finite quadratic module.
($\mathcal{O}$ = ring of integers of K)

Due to a result of Wall / Kneser, this finite quadratic module is isomorphic to a discriminant module D_L of an even integral lattice (i.e. [1]) ~~with signature~~ $s = \mathrm{signature}(L)$) ~~s.~~ Accordingly, by Milgram's formula[2], they have

$$B(\omega) = e\left(\frac{s}{8}\right).$$

[1] there exist a lattice $\underline{L} = (L, B)$ with dual lattice $L^\#$, such that $(L^\#/L, \ x + L \mapsto \frac{1}{2}B(x,x) + \mathbb{Z}) \simeq \underline{M}_\omega$. [2] Milgram's formula: $\sum_{x \in L^\#/L} e(\frac{1}{2}B(x,x)) = \sqrt{|L^\#/L|}\, e_8(s)$.

However I will not make use of these facts. Because it is difficult to find the discriminant module which is isomorphic to the one we have and hence the signature.

§1. Hecke's Reciprocity Formula:

The so far known deep result about the $C(\omega)$ is Hecke's reciprocity law.

By using the renormalized Hecke Gauss sums $B(\omega)$, it can be reformulated in the following way:

Thm 1: (Reformulation of the Hecke reciprocity formula)

For any $\omega \in K^*$, one has

$$B(\omega) = e\left(\frac{\mathrm{signature}(\omega)}{8}\right) B\left(\frac{-\delta^2}{4\omega}\right). \quad (1)$$

Here δ denotes any number in K s.t. $\delta\partial$ is integral and relatively prime to [redacted] $(4\omega\partial)$.
(i.e.: den. of $\frac{\delta^2}{4\omega}\partial$)

Moreover, $\mathrm{signature}(\omega) = \sum_{\sigma : K \hookrightarrow \mathbb{R}} \mathrm{sgn}(\sigma(\omega))$, ~~where~~

Pf: (Skoruppa, B) Assume K has class number 1.
Hecke's original Pf uses theta functions and it is complicated.

We give a short and elementary proof.

Using the obvious identity $\overline{B(\omega)} = B(-\omega)$, we can write the reciprocity formula in (1) as

$$B(\omega)\; B\!\left(\frac{\partial^2}{4\omega}\right) = e\left(\text{signat}\cancel{\text{ure}\left(\tfrac{\omega}{\delta}\right)}\right) \qquad -- \quad (2)$$

$\partial = \delta\mathcal{O}$. Then $\omega\delta = \frac{\beta}{\alpha}$, with relatively prime integers α, β in K.

Let us take $\gamma = \frac{1}{\delta}$.

$$\frac{\partial^2}{4\omega}\partial = \frac{1}{4\omega\delta} = \frac{\alpha}{4\beta}$$

Then the LHS of (2) becomes

$$\frac{1}{\sqrt{N(2\alpha\beta)}} \sum_{\substack{\mu \bmod \alpha \\ \nu \bmod 2\beta}} e\left(tr\left(\frac{\mu^2\beta}{\alpha\delta} + \frac{\nu^2\alpha}{4\beta\delta}\right)\right), \quad \text{provided}$$

$$\frac{\mu^2}{4\omega} = \frac{\gamma^2\alpha}{4\delta^2\beta}$$

~~Let us assume the case~~ where α is odd. Then 4β ~~bea~~ is the exact denominator of $\frac{\partial^2}{4\omega}\partial = \frac{\alpha}{4\beta}$.

We write

$$\frac{\mu^2\beta}{\alpha\delta} + \frac{\nu^2\alpha}{4\beta\delta} \equiv \frac{(2\mu\beta + \nu\alpha)^2}{4\beta\alpha\delta} \quad \bmod \frac{1}{\delta}\mathcal{O}$$

~~So, written~~

We see that $(\mu,\nu) \longmapsto 2\mu\beta + \nu\alpha$ defines an isomorphism of

$$\mathcal{O}/\alpha\mathcal{O} \times \mathcal{O}/2\beta\mathcal{O} \simeq \mathcal{O}/2\alpha\beta\mathcal{O}$$

Then LHS of (2) becomes

$$\frac{1}{\sqrt{N(2\alpha\beta)}} \sum_{\tau \bmod 2\alpha\beta} e\left(tr\left(\frac{\tau^2}{4\alpha\beta\delta}\right)\right).$$

Now, consider the lattice $L = (\mathcal{O}, B)$, where (4)

B is the bilinear form on $\mathcal{O}$ defined by

$B(x,y) = \frac{tr(2\alpha\beta xy)}{\delta}$. Easy to see that B is

nondegenerate and takes on even integral values. Then,

$$\mathcal{O}^{\#} = \{ y \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O} \mid B(y, \mathcal{O}) \subseteq \mathbb{Z} \} = \frac{1}{2\alpha\beta}\mathcal{O}$$

dual of $\mathcal{O}$ w.r.t. B

So, with these notations, the last sum can be

written as

$$\frac{1}{\sqrt{|\mathcal{O}^{\#}/\mathcal{O}|}} \sum_{x \in \mathcal{O}^{\#}/\mathcal{O}} e\left(\tfrac{1}{2} B(x,x)\right) = e\left(\frac{s}{8}\right)$$

Millgram's formula

Here s is the signature of the quadratic form $B(x,x)$

on $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}$. It is easy to see $s = \text{signature}(\omega)$. ∎

(for totally real number fields)

§2. Explicit Formula for the Hecke Gauss Sums for Quadratic Number Fields:

For quadratic number fields we ~~solved the~~ ~~open problem~~

of ~~I~~ obtained an explicit formula for

$C(\omega)$. For general number fields --

Let $K = \mathbb{Q}(\sqrt{D})$ with D: disc. of K.

From now on and we set:

$$G(\omega) := \sum_{\mu \bmod \mathfrak{a}} e\left(tr\left(\frac{\mu^2\omega}{\sqrt{D}}\right)\right)$$

Note: $G(\omega\sqrt{D}) = C(\omega)$.

Thm 2: (B) Let M be the smallest positive rational integer s.t. $M\omega$ is integral. Then,

$$G(\omega)^{B(\omega/\sqrt{D})} \Big/ N(\mathfrak{a}\cdot(u,2)) = \frac{1}{\sqrt{u}} \sum_{\Delta_1} \left(\frac{\Delta_1}{A}\right)\left(\frac{4N\beta/\Delta_1}{N/|\Delta_1|}\right)\sqrt{\operatorname{sgn}(\Delta_1)}$$

Here $\beta := M\omega$ and $N = M$ unless D, M are even and $M\omega \equiv M\omega' \bmod 2\sqrt{D}$; in the latter case $\beta = M\omega/2$ and $N = M/2$.

The sum is over all integers Δ_1 s.t. $|\Delta_1| = \gcd(N, 4N\beta)$ and $\Delta_1, 4N\beta/\Delta_1 \equiv 0, 1 \bmod 4$.

A denotes any integer relatively prime to N and s.t. $A = \operatorname{tr}\left(\frac{\beta\mu^2}{\sqrt{D}}\right)$ for some $\mu \in \mathcal{O}$.

Finally, $u = 2$ if there are 2 terms in the sum and 1 otherwise.

Note that there are at most 2 terms in the sum.

Note that $G(\omega)^{B(\omega/\sqrt{D})} / N(\mathfrak{a}\cdot(u,2))$ is an eighth root of unity if $B(\omega/\sqrt{D}) \neq 0$.

Sketch of proof of Thm 2:

The lefthand side defines a function $G: K^* \to \mathbb{C}$ ~~with the following properties~~ and ~~satisfies~~: we know the following about G (calculated by me)

a) (B) Explicit formula for denominator a prime ideal.

b) Product formula ($G(\omega) = \prod G(\omega_i)$, ω_i has prime ideal power denominator)

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c) Prime power reduction formula

$G(\omega) = G(\omega')N(p)$ with ω having denominator p^n and

w' having denominator p^{n-2} (for p odd). ($n \geq 2$)
(for even prime ideal power denominator, we have also a similar reduction formula)

(a)-(c) ~~define~~ uniquely determines G.
we call the fn. on rhs of the formula as F. Also,
$F : K^* \hookrightarrow \mathbb{C}$ and satisfies the above properties.

~~For~~ totally real # fields it is easy.

To compute the signature of the form we need ~~the~~ gram matrix ~~which can be~~ and ... and we have to diagonalise it.

$$\Delta^t D \Delta = G$$

$$\Delta = \begin{pmatrix} \alpha_1 & \cdots & \alpha_n \\ \sigma_2(\alpha_1) & \cdots & \sigma_n(\alpha_n) \\ \vdots & & \\ \sigma_n(\alpha_1) & \cdots & \sigma_n(\alpha_n) \end{pmatrix}$$

$$D = \begin{pmatrix} \omega & & & \\ & \sigma_2(\omega) & & \\ & & \ddots & \\ & & & \sigma_n(\omega) \end{pmatrix}$$

$$\tau_a(b) = \sum_{k=1}^{a} \exp\left(\frac{2\pi i k^2 b}{a}\right)$$

$$\tau_a(1) = \begin{cases} i\sqrt{a} & \text{if } a \equiv 3\ (4) \\ \sqrt{a} & \text{if } a \equiv 1\ (4) \end{cases}$$

Let p, q be differnt odd primes. Then using ~~the~~ result $\tau_p(q)\,\tau_q(p) = \tau_{pq}(1)$ we get

& $\tau_p(q) = \left(\frac{q}{p}\right)\tau_p(1)$

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)\tau_p(1)\,\tau_q(1) = \tau_{pq}(1)$$

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)\sqrt{p}\sqrt{q}\sqrt{\left(\frac{-4}{p}\right)}\sqrt{\left(\frac{-4}{q}\right)} = \sqrt{pq}\sqrt{\left(\frac{-4}{pq}\right)}$$

So, $\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if at least one of } p \text{ or } q \equiv 1\ (4) \\ -1 & \text{otherwise} \end{cases}$

$K = \mathbb{Q}$

$C(\omega)$

$\omega = \frac{p}{q}$

$$C\left(\frac{p}{q}\right) = C\left(\frac{-q}{4p}\right)\cdot e_8\left(\frac{\ }{4}\right)$$

$$\left(\frac{p}{q}\right)C\left(\frac{1}{q}\right) = \left(\frac{-4}{1}\right) C\left(\frac{-4q}{p}\right) C\left(\frac{-pq}{4}\right)$$

$$\left(\frac{p}{q}\right)C\left(\frac{1}{q}\right) = \left(\frac{-4q}{p}\right) C\left(\frac{1}{p}\right) C\left(-\frac{pq}{4}\right)$$

$$C\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right) C\left(\frac{1}{q}\right) \overset{HRL}{=} C\left(\frac{-q}{4}\right) e_8(1)\left(\frac{p}{q}\right)$$

$$= \left(\frac{p}{q}\right)\left(\frac{-4}{q}\right) C\left(\frac{-1}{4}\right) e_8(1)$$

$$C\left(\frac{q}{p}\right) = \left(\frac{q}{p}\right)\left(\frac{-4}{p}\right) C\left(\frac{-1}{4}\right) e_8(1)$$