Lille - 03-12-2008

Hilbert-Jacobi Forms of Critical Weight on the ~ Full Modular Group ~

We have as a simplest Jacobi form,

$$\vartheta(\tau,z) = q^{\frac{1}{8}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n\geq 1} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1}) \in J_{\frac{1}{2},\frac{1}{2}}(\varepsilon^3)$$

Recenty, Prof. Gritsenko, Skoruppa, Zagier discovered that many Jacobi forms can be constructed from ϑ.

<u>Goal</u>: Is there anything similar for Jacobi forms over totally real number fields?

Let $[K:\mathbb{Q}] = n \geq 1$, $\sigma_i : K \hookrightarrow \mathbb{R}$, $i=1,\dots,n$, K totally real,

$m \gg 0$, $m \in \mathfrak{d}_K^{-1}$ ($\mathfrak{d}_K^{-1} = \{x \in K \mid tr(xy) \in \mathbb{Z},\ \forall y \in \mathfrak{O}_K\}$).

Firstly, we define the double cover of $SL(2,\mathbb{Z}_K)$. ($\mathbb{Z}_K \to$ ring of integers of K.)

$Mp(2,\mathbb{Z}_K) = \{ (A, w_A) \mid A \in SL(2,\mathbb{Z}_K),\ w_A : \mathfrak{h}^n \xrightarrow{hol} \mathbb{C},\ w_A^2(\tau_1..\tau_n) = \prod_{j=1}^{n} (\sigma_j(c)\tau_j + \sigma_j(d)) \}$

[Multiplication in $Mp(2,\mathbb{Z})$ is given by
$(A, w_A)(B, w_B) = (AB, w_A(\sigma_1(B)\tau_1, \dots \sigma_n(B)\tau_n) w_B(\tau_1, \dots \tau_n))$]

The space of Jacobi forms:

<u>Defn</u>: Jacobi forms over (totally real) # fields with character on $Mp(2,\mathbb{Z}_K)$ denoted by $J_{k,m}(\chi)$ (with weight k, index m) consists of the following functions:

$\phi : \mathfrak{h}^n \times \mathbb{C}^n \to \mathbb{C}$ holomorphic ($\mathfrak{h}$: upper half plane) satisfying:

(1) $\phi|_{k,m} (A, w_A)(\vec{\tau}; \vec{z}) = w_A(\vec{\tau})^{-2k}\, e\left(-\frac{mc^2 z}{c\tau + d}\right) \phi\left(\dots \sigma_i(A)\tau_i; \dots, \frac{z_i}{\sigma_i(c)+\sigma_i(d)}, \dots\right)$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}_K)$

$= \phi(\vec{\tau}, \vec{z})\ \chi((A, w_A))$

(2) $\phi|_{k,m} [\lambda, \mu](\vec{\tau}, \vec{z}) = e(-(\lambda^2 m\tau + 2m\lambda z))\ \phi(\vec{\tau}, \vec{z} + \lambda\vec{\tau} + \mu) = \phi(\vec{\tau}, \vec{z})$, $\quad \lambda, \mu \in \mathfrak{O}_K$

where, ②

$$\sigma_i(A)\tau_i := \frac{\sigma_i(a)\tau_i + \sigma_i(b)}{\sigma_i(c)\tau_i + \sigma_i(d)}, \quad (\vec{\tau};\vec{z}) := (\tau_1 .. \tau_n ; z_1 .. z_n)$$

$$tr(\alpha z) := \sum_{i=1}^{n} \sigma_i(\alpha) z_i, \quad N(\alpha z) = \prod_{i=1}^{n} \sigma_i(\alpha) z_i, \quad e_\beta(\alpha z) = e^{2\pi i\, tr\left(\frac{\alpha z}{\beta}\right)}$$

By (1) & (2), we can see that

$$\underset{(\tau_1+\sigma_1(\alpha), \ldots \tau_n+\sigma_n(\alpha))}{\underset{\shortparallel}{\phi(\vec{\tau}+\alpha, \vec{z})}} = \phi(\vec{\tau};\vec{z}), \quad \phi(\vec{\tau}, \vec{z}+\alpha) = \phi(\vec{\tau}, \vec{z})$$

where $\sigma_i(A)\tau_i := \frac{\sigma_i(a)\tau_i+\sigma_i(b)}{\sigma_i(c)\tau_i+\sigma_i(d)}$, $(\vec{\tau};\vec{z}) = (\tau_1\ldots\tau_n;z_1,\ldots z_n)$ (2)

$tr(\alpha z) := \sum_{i=1}^{n}\sigma_i(\alpha)z_i$, $\mathbb{e}_\beta(\alpha z) := e^{2\pi i\, tr(\frac{\alpha}{\beta}z)}$.

By (1) & (2), ϕ satisfy a certain periodicity condition, namely we have

$$\phi(\vec{\tau}+\alpha;\vec{z}) = \phi(\vec{\tau},\vec{z}), \quad \phi(\vec{\tau},\vec{z}+\alpha) = \phi(\vec{\tau},\vec{z})$$

$\downarrow$

$(\tau_1+\sigma_1(\alpha),\ldots \tau_n+\sigma_n(\alpha))$

Since ϕ is holomorphic,

$$\phi(\vec{\tau};\vec{z}) = \sum_{n,r\in\mathfrak{d}_K^{-1}} c(n,r)\,\mathbb{e}(n\vec{\tau}+r\vec{z})$$

$\downarrow$ simple fn. satisfying the above periodicity condition

- Since $K\neq\mathbb{Q}$, one has in addition that

$c(n,r)=0$ unless $4mn-r^2 \geqslant 0$ ($\Leftrightarrow n \gg 0$)

(Koecher Principle)

- But for $K=\mathbb{Q}$, we have to add this condition as an axiom (3) to the defition of Jacobi forms.

After making suitable substitions such as

$D=-4mn+r^2$ and writing, we get ($C(D,r) = c(\frac{r^2-D}{4m},r)$)

$$\phi(\vec{\tau},\vec{z}) = \sum_{\substack{r\in\mathfrak{d}_K^{-1}\\ \frac{r^2-D}{4m}\in\mathfrak{d}_K^{-1}}} C(D,r)\,\mathbb{e}\left(\frac{r^2-D}{4m}\vec{\tau}+r\vec{z}\right)$$

Using (2), we obtain

$C(D,r) = C(D, r+2m\lambda)$ $(\forall\lambda\in\mathfrak{O}_K)$

So, this means $C(D,r)$ depends only on r mod $2m\,\mathfrak{O}_K$ =

• Then after dividing the sum into sums, we get that

(3) $\Phi(\vec{\tau},\vec{z}) = \sum_{\rho \bmod 2m\mathbb{Z}_K} h_\rho(\vec{\tau})\, \vartheta_{m,\rho}(\vec{\tau},\vec{z})$, where

$$\vartheta_{m,\rho}(\vec{\tau},\vec{z}) = \sum_{\substack{r \in \partial_K^{-1} \\ r \equiv \rho \bmod 2m}} e\left(\frac{r^2}{4m}\vec{\tau} + r\vec{z}\right).$$

• Now if we apply $|_{k,m}$ both sides of (3), we get

(4) $\Phi|_{k,m}\, \alpha\cdot(\vec{\tau},\vec{z}) = \sum_\rho h_\rho |_{k,m}\, \alpha\cdot\vec{z}\ \ \underbrace{\vartheta_{m,\rho}}_{\text{has weight } \frac{1}{2}}|_{\frac{1}{2},m}\, \alpha\cdot(\vec{\tau},\vec{z})$ ← transformation law for theta series

$\parallel$

$\chi(\alpha) \sum h_\rho(\vec{\tau})\vartheta_{m,\rho}$

<u>Thm:</u> $\vartheta_{m,\rho}|_{\frac{1}{2},m}\, \alpha = \vartheta_{m,\rho}$ $\forall \alpha \in \Gamma(4m\partial_K)^*$. (Here, $\Gamma(4m\partial_K)^*$ is a certain subgroup of $Mp(2,\mathbb{Z}_K)$, whose proj. to its first coordinate equals $\Gamma(4m\partial_K)$, <u>i.e</u> : the subgroup of all $A \in SL(2,\mathbb{Z}_K)$ with $A \equiv \mathbb{1} \bmod 4m\partial_K$.)

Then, (4) becomes

$$\chi(\alpha) \sum h_\rho(\vec{\tau})\vartheta_{m,\rho} = h_\rho|_{k-\frac{1}{2},m}\, \alpha\ \ \vartheta_{m,\rho}$$

So, $\chi(\alpha) h_\rho(\vec{\tau}) = h_\rho|_{k-\frac{1}{2},m}\, \alpha$. This implies that

$h_\rho \in M_{k-\frac{1}{2}}(\Gamma(4m\partial_K)^*, \chi)$.

From now on fix $k=\frac{1}{2}$.

Then

$h_\rho \in M_0(\Gamma(4m\partial_K)^*, \chi)$, so $h_\rho \equiv$ constant.

Now, lets define,

$Th_m = span_{\mathbb{C}} \langle \vartheta_{m,\rho} \mid \rho \in \partial_K^{-1}/2m\mathbb{Z}_K \rangle$

$\dim_{\mathbb{C}} Th_m = N(2m) \cdot D_K$
↳ discriminant of K.

The space Th_m is an $Mp(2,\mathbb{Z}_K)$-module wrt $|_{\frac{1}{2},m}$:

Follows from the following transformation for 3 type of matrices which generate $SL(2,\mathcal{O}_K)$, s.t.

$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, $\varepsilon \in \mathcal{O}_K^*$, $B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathcal{O}_K$, $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(1) $\vartheta_{m,\rho} |_{\frac{1}{2},m} (A,w_A)(\vec{\tau},\vec{z}) = \sqrt{N\varepsilon}\, \vartheta_{m,\varepsilon\rho}$

(2) $\vartheta_{m,\rho} |_{\frac{1}{2},m} (B,w_B)(\vec{\tau},\vec{z}) = e\left(\frac{\rho^2 b}{4m}\right) \vartheta_{m,\rho}$

> directly from definition

(3) $\vartheta_{m,\rho} |_{\frac{1}{2},m} (C,w_C)(\vec{\tau},\vec{z}) = D_K^{-\frac{1}{2}}\, e_8(-n)\, N(2m)^{-\frac{1}{2}} \sum_{r \in \partial_K^{-1}/2m} e_{2m}(-\rho r)\, \vartheta_{m,r}(\vec{\tau},\vec{z})$

↳ uses Poisson summation formula.

Hence,

$J_{\frac{1}{2},m}(\chi) = Th_m^{Mp(2,\mathbb{Z}_K),\chi}$, where

$Th_m^{Mp(2,\mathcal{O}_K),\chi} = \{ \vartheta \in Th_m \mid \vartheta|_{\frac{1}{2},m} \tilde{A} = \chi(\tilde{A})\,\vartheta,\ \forall A \in Mp(2,\mathbb{Z}_K) \}$

Now, we study the decompositions of Th_m. (5)

as $Mp(2, \mathbb{Z}/k)$ - modules and find the 1 dim

$Mp(2, \mathcal{O}_K)$ - submodules, since

$$J_{\frac{1}{2}, m}(\chi) \hookrightarrow \begin{array}{c} \text{1-dim} \\ \text{submodules} \\ \text{of } Th_m \end{array}$$

For this, we need to define some concepts such as Th_m^{old}, Th_m^{new}, G_m, $Th_m^{new, \chi}$.

Thm: Let $U_d : Th_m \to Th_{md^2}$

a) $\phi|_{U_d}(\vec{z}, \vec{z}) = \phi(\vec{z}, d\vec{z})$, $d \in \mathcal{O}_K$

b) $Th_m \to Th_{md^2}$, $v \mapsto v|U_d$ defines an injective $Mp(2, \mathbb{Z}/k)$ - hom. $SL(2, \mathcal{O}_K)$ - hom.

Now, define

$$Th_m^{old} := \sum_{\substack{d^2 m' = m \mathcal{O}_K \\ d \in \mathcal{O}_K \\ d \text{ not a unit}}} Th_{m'} | U_d$$

Th_m^{new} := orthogonal complement of Th_m^{old}.

Remark: We define the scalar product over Th_m as:

$$v = \sum_{\rho \in \partial^{-1}/2m} c_{1,\rho} v_{m,\rho}, \quad v' = \sum_{\rho' \in \partial^{-1}/2m} c_{2,\rho} v_{m,\rho'},$$

$$\langle v, v' \rangle = \sum_{\rho=1}^{N(2m) D_K} c_{1,\rho} \overline{c_{2,\rho}}$$

Then it can be shown easily that $Mp(2, \mathcal{O}_K)$ - action is unitary w.r.t. $\langle , \rangle$ (3 type of matrices)

Hence, using this we can show that Th_m^{new} is also an $Mp(2, \mathcal{O}_K)$ module of Th_m. (3)

Now define

$$G_m = \left\{ \varepsilon \in \mathcal{O}_K / 2m\mathfrak{d}_K \;\middle|\; \varepsilon^2 \equiv 1 \; 4m\mathfrak{d}_K \right\}$$

<u>Remark:</u> Let $\underline{M}_m = \left(\mathfrak{d}_K^{-1}/2m\mathcal{O}^2, \; tr\left(\frac{x^2}{4m}\right) \right)$ be a f.q.m.
Then, G_m is the subgroup of all (elements of the $O(\underline{M}_m)$ which are $\mathcal{O}_K$-module homomorphisms.
(in fact Th_m is Weil Repr. associated to M_m)

The action of G_m on Th_m given as,
$(\vartheta_{m,\rho}, \varepsilon) \longmapsto \vartheta_{m, \varepsilon\rho}$ for $\varepsilon \in G_m$.
Then the space,

$$Th_m^{new, \chi} = \left\{ \vartheta \in Th_m^{new} \;\middle|\; \varepsilon\vartheta = \chi(\varepsilon)\vartheta, \; \forall \varepsilon \in G_m \right\}$$

is also an $Mp(2, \mathbb{Z}_K)$-module, since the G_m action commutes with the $Mp(2, \mathbb{Z}_K)$-action.

Now, it is easy to deduce

<u>Thm:</u>

$$Th_m = \bigoplus_{\substack{d, m' \\ d^2 m' = m \\ d \in \mathbb{Z}_K, \\ d: \text{not unit}}} \bigoplus_{\chi \in \widehat{G}_{m'}} U_d\left(Th_{m'}^{new, \chi}\right)$$

→ irreducible, none of them zero dim.

→ $Mp(2, \mathcal{O}_K)$ invariant

$\widehat{G}_{m'}$ = character group of G_m

<u>Thm:</u> # irreducible components in $Th_m \leq \sigma(m\mathfrak{d}_K)$

$$= \sum_{t \mid m\mathfrak{d}_K} 1$$

Recall we are interested in 1-dim. subspaces since (7)

$J_{\frac{1}{2},m}(\chi) \longleftrightarrow$ 1 dim irreducible subspaces of Th_m

From now on, we restrict to the case,

K : quadratic # field, (recall, totally real)

$m\mathfrak{d}_K$: squarefree ($Th_m^{new} = Th_m$)

D_K : odd, square-free ($D_K > 0$)

Let ε be the fundamental unit of K, $\varepsilon > 1$.

Remark: Every character of G_m is of the form

$$\chi_f(\varepsilon) = \mu_f\left(\frac{\varepsilon+1}{2}\right), \quad f \mid m\mathfrak{d}_K, \quad \varepsilon \in G_m$$

$$\mu_f\left(\frac{\varepsilon+1}{2}\right) = (-1)^{\#\{\mathfrak{p} \mid \mathfrak{p} \mid f \ \& \ \mathfrak{p} \mid \frac{\varepsilon+1}{2}\}} \quad (\mathfrak{p} \text{ prime ideal})$$

Thm: For $f \mid m\mathfrak{d}_K$,

$$\dim Th_m^{\chi_f} = \frac{2^{n}}{2^{\#\text{prime ideals dividing } m\mathfrak{d}_K}} \prod_{\mathfrak{p} \mid m\mathfrak{d}_K} \left(N(\mathfrak{p}) + \mu_f(\mathfrak{p})\right)$$

(All factors are integer)

Main Thm: $J_{\frac{1}{2},m}(\chi) = 0$ for all $m \in \mathfrak{d}_K^{-1}$, all characters χ of $Mp(2,\mathcal{O}_K)$ except for the following cases:

Case 1: $N(\varepsilon) = -1$, 2 splits completely, $m = \frac{2}{\omega}$, $\omega = \varepsilon\sqrt{D_K}$

Then $J_{\frac{1}{2},m}(\chi) = \mathbb{C}\vartheta^k$, where χ is a character afforded by ϑ_1^k and

$$\vartheta^k(\vec{\tau},\vec{z}) = \sum_{r \in \mathfrak{O}_K} \left(\frac{-4}{N(r)}\right) e\left(\frac{\vec{\tau} r^2}{8\omega} + \frac{r}{\omega}\vec{z}\right)$$

Case 2 : $N(\varepsilon) = -1$, 2 splits completely, 3 not inert, $m = 6/\omega$, where

$$J_{\frac{1}{2},m}(X) = \mathbb{C}\vartheta^k, \text{ where}$$

$$\vartheta_3^k(\vec{\tau},\vec{z}) = \sum_{r \in \mathfrak{O}_K} \left(\frac{12}{N(r)}\right) e\left(\frac{\vec{\tau} r^2}{24\omega} + \frac{r}{\omega}\vec{z}\right)$$

Case 3: $N(\varepsilon) = -1$, 2 splits completely, 3 not inert, $3 = \pi_1 \cdot \pi_2$ with $\pi_1, \pi_2 \gg 0$, $m = 2\pi_1/\omega$

$$J_{\frac{1}{2},m}(X) = \mathbb{C}\,\vartheta_{\pi_1}^k, \text{ where}$$

$$\vartheta_{\pi_1}^k(\vec{\tau},\vec{z}) = \sum_{r \in \mathfrak{O}_K} \left(\frac{-4}{N(r)}\right)\left(\frac{r}{\pi_1}\right) e\left(\frac{\tau r^2}{8\pi_1\omega} + \frac{r}{z}\omega\right)$$

Here,

$$\left(\frac{r}{\pi_1}\right) = \begin{cases} +1 & r \equiv \square \bmod \pi_1 \\ -1 & r \not\equiv \square \quad '' \\ 0 & \text{otherwise} \end{cases}$$

Case 1: 11% (over all squarefree, odd D_K)

Case 2: 4%

Case 3: 4%

$\Big(N(\varepsilon) = -1$: $K = \mathbb{Q}(\sqrt{p})$, $p \equiv 1(4)$, $N\varepsilon = -1\Big)$

Plan: 1) $[K:\mathbb{Q}] > 2$

2) ϑ^k posses Π-expansion

3) Critical weight.