→ Theta Expansion proof

§2. Lecture 2 - Chennai - 01-12-2009

Connection Between Jacobi Forms over NF and VVMF:

We have to look into these $\vartheta_{m,x}$ more closely.

We introduce

$$Th_m := \mathrm{span}_{\mathbb{C}} \langle \vartheta_{m,x} \mid x \in \mathfrak{d}^{-1}/2m\mathcal{O} \rangle .$$

It turns out that $SL(2,\mathcal{O})$ acts on Th_m via $|_{\frac{1}{2},m}$

Actually, it is not really a real action, ~~so we~~ it is a projective one. To have a really an action we need to define the metaplectic cover of $SL(2,\mathcal{O})$.

It is denoted by $Mp(2,\mathcal{O})$ and defined by

~~Mp(2,O) = metaple~~

$$Mp(2,\mathcal{O}) = \{ (A,w) \mid w \in \mathfrak{h}^n \xrightarrow{hol} \mathbb{C} , w^2(\tau) = N(c\tau+d) \} .$$

Composition law:

$$(A,w)(B,v) = (AB, w(B\tau)v(\tau)) .$$

For $k \in \frac{1}{2}\mathbb{Z}$, $Mp(2,\mathcal{O})$ acts on $Hol(\mathfrak{h}^n)$ and $Hol(\mathfrak{h}^n \times \mathbb{C}^n)$ via $|_k$ resp. $|_{k,m}$, where $|_k$ and $|_{k,m}$ are defined by the same formulas as before but with $N(c\tau+d)^{-k}$ replaced by w^{-2k}.

Thm: Th_m is left invariant under the $|_{\frac{1}{2},m}$ action of $Mp(2,\mathcal{O})$.

Pf: We know $SL(2,\mathcal{O})$ is generated by

$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ~~and $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$, for $\varepsilon \in \mathcal{O}^*$.~~

due to Vaserstein 1972   6GOK

~~But~~

$$\cancel{\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix} = \dots}$$

~~So, actually $SL(2,\mathcal{O})$ is~~

Using this fact it is can be deduced that $M_p(2,\mathcal{O})$ is generated by $\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right)$ and $\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \sqrt{\tau}\right)$, where always take $\sqrt{\tau} \in$

To prove the theorem, it is enough to prove for these generators. More specifically we have

i) $\vartheta_{m,x}\Big|_{\frac{1}{2},m}\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right) = e\left(\frac{x^2 b}{4m}\right)\vartheta_{m,x}$

ii) $\vartheta_{m,x}\Big|_{\frac{1}{2},m}\left(\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}, \sqrt{\tau}\right) = c\sum_{y\in\partial^{-1}/2m\mathcal{O}} e\left(\frac{-xy}{2m}\right)\vartheta_{m,y}$,

where $c = \frac{1}{|\partial^{-1}/2m\mathcal{O}|}\sum_{x\in\partial^{-1}/2m\mathcal{O}} e\left(-\frac{x^2}{4m}\right)$.

We show (i). For (ii) we need Poisson summation formula, it is a lengthy calculation. So, I skip this.

$$\vartheta_{m,x}\Big|_{\frac{1}{2}m}\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right)(\tau,z) = \vartheta_{m,x}(\tau+b, z)$$

$$= \sum_{r\equiv x \bmod 2m\mathcal{O}} q^{\frac{r^2}{4m}}\zeta^r \cdot e\left(\frac{r^2 b}{4m}\right)$$

$$= e\left(\frac{x^2 b}{4m}\right)\vartheta_{m,x}$$

Now we use this result to define a repre-sentation of $Mp(2,\mathcal{O})$.
Let $V = \mathbb{C}[\partial^{-1}/2m\mathcal{O}]$ ~~Here~~ be (vector space of formal linear combinations of elements of the group $\partial^{-1}/2m\mathcal{O}$). We ~~denote~~ use e_x for the elements associated to $x \in \partial^{-1}/2m\mathcal{O}$.
Define

$$\rho : Mp(2,\mathcal{O}) \xrightarrow{Gl} (V),$$

$$\rho(\alpha)(e_x) = \sum_y \rho(\alpha)_{y,x}\, e_y \quad \text{where} \quad \rho(\alpha)_{y,x}$$

are defined by $\vartheta_{m,x}\, |_{\frac{1}{2},m}\, \alpha = \sum \rho(\alpha)_{x,y} \vartheta_{m,y}$.

<u>Theorem:</u> ρ is indeed a representation.
<u>Pf:</u> Straightforward.
<u>Remark:</u> ρ is the so-called Weil representation of $Sl(2,\mathcal{O})$ associated to the fqm $\left(\partial^{-1}/2m\mathcal{O}, tr\left(\frac{x^2}{4m}\right)\right)$.

<u>Thm:</u> ρ is unitary w.r.t to the scalar product

$$\langle e_x | e_y \rangle = \begin{cases} 1 & x=y \\ 0 & \text{otherwise} \end{cases}.$$

<u>Pf:</u> Straightforward calculation ~~of the~~ for the generators using (i) & (ii).

We have a transformation law for a Jacobi form ϕ ~~under the action of $SL(2,\mathbb{Z})$~~ w.r.t $SL(2,\mathbb{Z})$. Furthermore, we have the theta expansion and the transformation law for $\vartheta_{m,x}$. This should imply also that h_x have some transformation law.

To investigate it more closely, we introduce the notion of VVMF.

<u>Defn:</u> Let $\rho : Mp(2,\mathbb{Z}) \longrightarrow GL(V)$ be a finite dimensional representation, where $\ker\rho \underset{\text{f.index}}{\subseteq} Mp(2,\mathbb{Z})$. We set for $k \in \frac{1}{2}\mathbb{Z}$,

$$M_k(\rho) := \left\{ F : \mathfrak{h}^n \longrightarrow \mathbb{C} \;\middle|\; F|_k \alpha = \rho(\alpha) F \right\}$$

For the case $k = \mathbb{Q}$ we need to add some condition.

<u>Thm:</u> The application

$$\phi = \sum h_x \vartheta_{m,x} \longmapsto h = \sum_x h_x e_x$$

defines an isomorphism

$$J^{K}_{k,m}(\chi) \longrightarrow M_{k-\frac{1}{2}}(\chi\rho^*)$$

, where ρ^* is the conjugate representation of ρ.

$$\left(\text{i.e: } \rho^*(\alpha)(e_x) = \sum_{y \in \mathfrak{d}^{-1}/2m\mathfrak{o}} \overline{\rho(\alpha)}_{y,x}\, e_y \right)$$

Pf.

~~Let ϕ~~

we have $\phi = \sum_x h_x v_{m,x}$.

~~Now $\phi\,\chi(\alpha) =$~~

Then

$$= \phi|_{k,m}\alpha = \sum_x h_x|_{k-\frac{1}{2}}\alpha \; v_{m,x}|_{\frac{1}{2},m}\alpha \quad , \quad \alpha \in Mp(2,\mathbb{Q}).$$

$$= \sum_x h_x|^{\alpha}_{k-\frac{1}{2}} \sum_y \rho(\alpha)_{x,y} v_{m,y}$$

$$= \sum_y \Big(\sum_x h_x|^{\alpha}_{k-\frac{1}{2}} \rho(\alpha)_{x,y} \Big) v_{m,y}$$

Then $\chi(\alpha) h_y = \sum_x h_x|_{k-\frac{1}{2}}\alpha \; \rho(\alpha)_{x,y}$.

Equivalently, $h_y|_{k-\frac{1}{2}}\alpha^{-1} = \sum_x h_x \, \rho(\alpha)_{x,y} \; \chi(\alpha^{-1})$

But then,

$$h|_{k-\frac{1}{2}}\alpha = \sum_x h_x|_{k-\frac{1}{2}}\alpha \; e_x$$

$$= \sum_x \Big(\sum_y h_y \rho(\alpha^{-1})_{y,x} \chi(\alpha) \Big) e_x$$

$$= \cancel{\sum_y h_y \sum_x \rho(\alpha^{-1}}$$

$$= \sum_y h_y \sum_x \rho(\alpha^{-1})_{y,x} \chi(\alpha) e_x$$

But ρ is unitary, hence $\rho(\alpha^{-1})_{y,x} = \rho^{*}(\alpha)_{x,y}$.

So,

$$h|_{k-\frac{1}{2}}\alpha = \sum_y h_y \left(\sum_x \rho^{*}(\alpha)_{x,y}\,\chi(\alpha)\, e_x \right)$$

$$= \sum_y h_y\ \chi(\alpha)\ \rho^{*}(\alpha)\,(e_y)$$

$$= \chi(\alpha)\ \rho^{*}(\alpha) \left(\sum_y h_y e_y \right)$$

$$= \chi(\alpha)\ \rho^{*}(\alpha)\ h = \chi\rho^{*}(\alpha)\ h.$$

So, h is a ~~HMF~~. VVMF. So the map is well-defined.

Corr 1: $h_x \in M_{\frac{k-1}{2}}(\ker(\chi\rho^{*}))$ for all x.

Vice versa, if $h = \sum h_x e_x$ in $M_k(\rho^{*})$ then $\phi := \sum h_x \otimes \theta_{m,x}$ is a JF by rev. the arg. The inj. follows since for fixed τ the fns $\theta_{m,x}(\tau, \cdot)$ are lin. ind. ~~(exe)~~

Corr 2: If $\phi \in J_{k,m}^{K}(\chi)$ then $C_\phi(t,r) = 0$ unless $4mt - r^2 \gg 0$.

Corr 3: $\dim(J_{k,m}^{K}) \leq \dim M_k(\rho) < \infty$.

Pf of Corollaries:

Proof of Corr. 1:

h is invariant under $\ker(\chi\rho^{*})$. Hence, the coordinate functions h_x are also invariant under $\ker(\chi\rho^{*})$.

Proof of Corr 2: The Koecher principle holds for h_x from Corr 1. (since it holds for HMF's)

But $$h_x = \sum_{\substack{D \\ \frac{r^2-D}{4m} \in \mathcal{U}^{\#}}} q^{-\frac{D}{4m}} \, c_\phi\left(\frac{r^2-D}{4m}, r\right).$$

So, $c_\phi\left(\frac{r^2-D}{4m}, r\right) = 0$ unless $D \leq 0$. $\square$

<u>Proof of Cor 3:</u>

Follows from

$$\begin{array}{ccccc} J_{k,m}^{k}(X) & \xrightarrow{\simeq} & M_{k-\frac{1}{2}}(\rho^{*}X) & \hookrightarrow & \bigoplus_{x \in \mathfrak{d}^{-1}/2m\mathfrak{o}} M_{k-\frac{1}{2}}(\mathrm{Ker}(x\rho^{*})) \\ \phi & \longmapsto & h = \sum_x h_x e_x & \longmapsto & (h_x)_{x \in \mathfrak{d}^{-1}/2m\mathfrak{o}} \end{array}$$

We know space of HMF have finite dimension. Hence,

$\dim J_{k,m}^{k}(X) < \infty$.