$H(\Omega)$ ~~leaves~~ acts Thm invariant.

Thursday

同济大学 教师备课用纸

Lecture 3 - Chennai - 03-12-2009 (1)

§4. Singular Jacobi forms over k:

Recall: ϕ is singular if $C_\phi(D,r)\neq 0$ only when $D=0$.

→ It turns out that singular Jacobi forms are those of weight $\frac{1}{2}$. PTO

Theorem: Let $\phi\in J_{k,m}(\chi)$, and $\phi\neq 0$. TFAE:

i) ϕ is singular

ii) $k=\frac{1}{2}$

iii) $\phi\in Th_m$

iv) $\phi\in Th_m^{Mp(2,O),\chi}$

(Notation: Let X be an $Mp(2,O)$-module, then (X is invariant under the $|_{\frac{1}{2},m}$ action of $Mp(2,O)$)

(~~If~~ $X^{Mp(2,O),\chi} := \{ v\in X \mid v|g = \chi(g)v,\ \forall g\in Mp(2,O)\}$.

Pf:

"(i) ⇒ (ii)" Suppose ϕ is singular, write $\phi=\sum_x h_x \vartheta_{m,x}$.

⇒ h_x is constant for all x (since $h_x=\sum_D q^{\frac{-D}{4m}} C_\phi(D,x)$).

Since h_x is a HMF, $k-\frac{1}{2}=0$. So, $k=\frac{1}{2}$.

"(ii) ⇒ (i)" Suppose $k=\frac{1}{2}$. Then, h_x is constant for all x, since h_x has weight 0.

So, $\phi=\sum_x \vartheta_{m,x} h_x = \sum_x h_x \sum_{r\equiv x \bmod 2mO} q^{\frac{r^2}{4m}}\zeta^r$

$$C_\phi(D,r)=\begin{cases} - & D=0\\ 0 & D\neq 0\end{cases}$$

So, ϕ is singular.

$\left(\phi=\sum_{D,r} C_\phi(D,r)\, q^{\frac{r^2-D}{4m}}\zeta^r \right)$

So, we have to shortly extend our ~~pro~~ theory.

<u>Defn:</u> Let $k \in \frac{1}{2}\mathbb{Z}$, $\chi : Mp(2,\mathbb{Z}) \to S^1$ a character with ~~finite~~, where $\ker \chi \underset{\text{fi}}{\subseteq} Mp(2,\mathbb{Z})$.

$$J_{k,m}^{K}(\chi) = \left\{ \phi : \mathbb{H} \times \mathbb{C} \xrightarrow{\text{hol.}} \mathbb{C} \;\middle|\; \phi|_{k,m}\gamma = \chi(\gamma)\phi \quad \forall \gamma \in Mp(2,\mathbb{Z}) \right\}$$

<u>Rem:</u> If $k \in \mathbb{Z}$ this coincides with the old def.

For such $\phi \in J_{k,m}(\chi)$, we also have theta expansion.

$$\phi(\tau,z) = \sum_{x} \vartheta_{m,x}(\tau,z)\, h_x(\tau).$$

Here, h_x is again HMF of weight $k-\frac{1}{2}$, which might be integral.

(ii) $\Leftrightarrow$ (iii) obvious.

"(ii) $\Rightarrow$ (iv)" $\forall \phi \in J_{k,m}$, ϕ is a linear combination of $\vartheta_{m,x}$ and ϕ is also a Jacobi form. So, $\phi \in J_{k,m}^{Mp(2,\mathcal{O}),\chi}$.

"(iv) $\Rightarrow$ (iii)" obvious.

$\blacksquare$

Remark: ϕ is a singular JF

~~($\Rightarrow$ $\mathbb{C}\phi$ is an $Mp(2,\mathcal{O})$-submodule of $J_{k,m}$ from (iv).)~~

~~Step~~ (iv) $\Rightarrow$ ($\mathbb{C}\phi \subseteq J_{k,m}$ is an $Mp(2,\mathcal{O})$-module).

If $\mathbb{C}\phi \subseteq J_{k,m}$ is an $Mp(2,\mathcal{O})$-module. This means

$\phi|\alpha = c(\alpha)\phi$, $\forall \alpha \in Mp(2,\mathcal{O})$.

Hence c must be a character of $Mp(2,\mathcal{O})$.

Hence, $\phi \in J_{k,m}^{Mp(2,\mathcal{O}),c}$, i.e. (iv) is satisfied. By the Thm ϕ is singular.

Hence, to determine all singular Jacobi forms we have to determine for each m, the 1-dim $Mp(2,\mathcal{O})$-submodules of $J_{k,m}$.

~~§5. Decomposition of $J_{k,m}$~~

Remark: Singular Jacobi forms for $K=\mathbb{Q}$ are precisely, the functions (Skoruppa 1985)

$\vartheta(\tau, dz)$, $\psi(\tau, dz)$ $d \in \mathbb{Z}_{\geq 1}$, where

$$\vartheta(\tau,z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^r \in J_{\frac{1}{2},2}(\varepsilon^3)$$

$$\psi(\tau,z) = \sum \left(\frac{12}{r}\right) q^{\frac{r^2}{24}} \zeta^r \in J_{\frac{1}{2},3}(\varepsilon^3)$$

$$\varepsilon(A,w) = \frac{\eta(A\tau)}{\eta(\tau)\, w(\tau)} \qquad \eta = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n)$$

$\vartheta(\tau, z)$ is the function which occurs in the Jacobi triple product identity and it also in essence in the weierstrass sigma function associated to the elliptic curve $\mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$. More precisely,

$$\vartheta(\tau,z) \underset{\text{J.t.p identity}}{=} q^{\frac{1}{8}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n\geq 1} (1-q^n)(1-q^n\zeta)(1-q^n\zeta^{-1}).$$

$\vartheta(\tau,z) = \eta^3\,\sigma(\tau,z)$, where $\sigma(\tau,z)$ weierstrass σ-function.

(weierstrass σ-function helps us to construct the function from its divisors.) $\Rightarrow$ The goal is to ~~find~~ similar result for arbitrary number fields.

§5. Decomposition of Th_m:

Our main purpose of ~~task~~ this section is to ~~Main~~ prove the following theorem:

main theorem: We have the following decomposition of Th_m into ~~the~~ irreducible $Mp(2,\mathcal{O})$-modules.

$$Th_m = \bigoplus_{\substack{I:=(d)\in P \\ I^2 \mid m\mathfrak{d}}} \bigoplus_{\chi \in \hat{O}_{\frac{m}{d^2}}} Th^{new,\chi}_{m/d^2} \mid U_d \qquad (*)$$

Explanation:
Some ~~facts~~:

- ~~Here~~ the operator U is defined ~~as~~ for $d\in\mathcal{O}$ as follows:

$$U_d : Th_m \longrightarrow Th_{md^2} \quad \left(\begin{matrix} d\in\mathcal{O}, & m\in\mathfrak{d}^{-1} \\ m\gg 0, & md^2\gg 0 \end{matrix}\right)$$

$(\varepsilon_m, \vartheta_{m,x}) \mapsto (\varepsilon.\vartheta_{m,x} = \vartheta_{m,\varepsilon x})$

$$\vartheta \longmapsto \vartheta \mid U_d(\tau,z) := \vartheta(\tau, dz)$$

$O_m = \{ \varepsilon_m \in (\mathcal{O}/_{2m\mathfrak{d}})^* \mid \varepsilon_m^2 \equiv 1 \mod 4m\mathfrak{d} \}$

since multiplication by an $\varepsilon_m \in O_m$ defines an isometry orthogonal transformation of $M_m = (\mathfrak{d}^{-1}/_{m\mathfrak{d}}, tr(\frac{x^2}{4m}))$ O_m becomes a subgroup of the orth. gr. of M_m

~~The space~~ The ~~spaces~~ ~~$Th_m \mid U_d$~~ ~~d^2~~ χ ~~are~~ to be defined to be

$$Th^{new,\chi}_{m/d^2} := \{ \vartheta \in Th_m^{new} \mid \varepsilon.\vartheta = \chi(\varepsilon)\vartheta, \ \forall \varepsilon \in O_m \}$$ ~~is defined~~

for ~~χ~~ ~~being~~ a linear character χ of O_m ($O_m \to S^1$)

P: semigroup of principal ideals in $\mathcal{O}$.

We shall show below ~~Note also~~ that the spaces $Th_{m/d^2} \mid U_d$ ~~does not~~ depends only on the ideal I, so the exp. $(*)$ is well-defined.

We set

$$Th_m^{old} := \sum_{\substack{I := (d) \in \mathcal{P} \\ I^2 \mid m\partial \\ I \neq \mathcal{O}}} Th_{m/d^2}^{} \mid U_d$$

Th_m^{new} is the orthogonal complement of Th_m^{old} w.r.t. to the inner product:

$$\langle v_1, v_2 \rangle = \sum_{x \in \partial^{-1}/2m\mathcal{O}} c_{1,x} \overline{c_{2,x}} \quad \text{, where}$$

$$v_i = \sum_{x \in \partial^{-1}/2m\mathcal{O}} c_{i,x} \vartheta_{m,x} \quad , i = 1, 2 .$$

Remark: The spaces Th_m^{old}, Th_m^{new}, $Th_m^{new,x}$, $x \in \hat{\mathcal{O}}_m$ are $Mp(2, \mathcal{O})$-submodules of Th_m.

~~For~~ since the proof of the theorem we need ~~some~~ a series of lemmas.

~~Lem 1~~ The action of $(2m\partial)^{-1} \cdot (2m\partial)^{-1}$ on Th_m. $\Rightarrow P \neq \mathcal{O}$

<u>Lemma 1</u>: Let $I := (d) \in \mathcal{P}$. Suppose $I^2 \mid m\partial$. Then the spaces $Th_{m/d^2} \mid U_d$ depends only on the ideal I. $\square$

<u>Lemma 2</u>: We have the following decomposition of Th_m:

$$Th_m = \sum_{I := (d) \in \mathcal{P}} Th_{m/d^2}^{new} \mid U_d \qquad (*)$$

<u>Remark</u>: We shall show in a moment that the sum $(*)$ is actually direct. $I^2 \mid m\partial$

<u>Pf</u>: Do induction on $N(m\partial)$. If we have $N(m\partial) = 1$, then since Th_m^{old} is empty, $Th_m = Th_m^{new}$.

So, $Th_m = Th_m^{new} \mid U_1 = Th_m$.

Suppose ~~$N(m\partial) \geq 1$~~. $(*)$ holds true for all m' with $N(m'\partial) < N(m\partial)$:

we can write

$$Th_m \underset{\substack{\uparrow \\ \text{by def. of } Th_m^{old}}}{=} Th_m^{old} \oplus Th_m^{new} \underset{\substack{\uparrow \\ \text{by defi of } Th_m^{old}}}{=} Th_m^{new} \oplus \sum_{\substack{I := (d) \in \mathcal{P} \\ I^2 \mid m\partial \\ I \neq \mathcal{O}}} Th_{m/d^2} \mid U_d$$

Thm: The application $(\vartheta \,\S\,(x,r)) \mapsto \vartheta|_{\frac{1}{2},m}(x,r)$ defines a right action of the gr. ~~Gm~~ =

$G_m := ((2m\partial)^{-1} \times (2m\partial)^{-1}) \cdot (2m\partial)^{-2}$ of $H(K)$

on the space Th_m. The ~~Abst~~ G_m-right module Th_m is irreducible.

Lemma: The subgroup $G_m := ((2m\partial)^{-1} \times (2m\partial)^{-1}) \cdot (2m\partial)^{-2}$ of $H(K)$ ~~acts on~~ leaves the space Th_m invariant.

Pf: Let $\lambda, \mu \in (2m\partial)^{-1}$ and $r' \in (2m\partial)^{-2}$.

Then,

$$\vartheta_{m,x}|_{\frac{1}{2},m}((\lambda,\mu),r')$$

$$= \sum_{r \equiv x \bmod 2m\partial} q^{\frac{r^2}{4m}} \zeta^r(z+\lambda\tau+\mu)\, e(m\lambda^2\tau + 2\lambda z + \lambda\mu + r')$$

$$= \sum_{r \equiv x \bmod 2m\partial} q^{\frac{(r+2m\lambda)^2}{4m}} \zeta^{r+2m\lambda}\, e[\mu(r+m\lambda)+mr'].$$

$$= e[\mu(x+m\lambda)+mr'] \cdot \vartheta_{m,\,x+2m\lambda}.$$

$\to$ since the action of $Mp(2,\mathcal{O})$ commutes with ~~the action of~~ ϑ, and $\hookrightarrow$ ~~Gm~~ is unitary and commutes with the action of G_m.

By induction hypothesis, if $N(d) > 1$, the $Thm_{/d^2}$ can be written as ~~a sum of subspaces. as given so,~~ follows:

$$Thm_{/d^2}|U_d = \sum_{\substack{J:=(c)\in P \\ J^2 \mid \frac{md}{I^2}}} Thm^{new}_{/d^2c^2}|U_c|U_d .$$

So, we get

$$Thm = Thm^{new} \oplus \sum_{\substack{I:=(d) \\ I^2 \mid md \\ I \neq \mathcal{O}}} \sum_{\substack{J:=(c)\in P \\ J^2 \mid \frac{md}{I^2}}} Thm^{new}_{/d^2c^2}|U_c|U_d = \sum_{\substack{I':=(t)\in P \\ I'^2 \mid md}} Thm^{new}_{/t^2}|U_t \qquad \square$$

To show that the sum is direct, we need another ~~result~~, lemma.

Lemma (without proof)

Let $I:=(d)\in P$. Assume $I^2 \mid md$. For $v \in Thm$, set

$$Tr_m^I v := N(I)^{-2} \sum_{y \in I^{-1}/\mathcal{O} \times I^{-1}/\mathcal{O}} v|_{\frac{1}{2},m}[y].$$

Lemma 1

Then, $Tr_m^I v$ does not depend on the choice of repr. y for $y \in I^{-1}/\mathcal{O} \times I^{-1}/\mathcal{O}$ and it defines an operator ~~on~~ $Thm \to Thm$. In fact, Tr_m^I is the orthogonal projection of Thm onto the subspace $Thm_{/d^2}|U_d$.

($P = Tr_m^I$, we need to show: i) $P^2 = P$
ii) P is Hermitian
iii) $Image(P) = Thm_{/d^2}|U_d$
iv) $Image(P)$ & $kernel(P)$ are orth. subspaces of Thm.)

(enough to show that $I^{-1} \subseteq (2md)^{-1}$
$I^2 \mid md$, $I \mid 2md$
$2md \subseteq I$, $I^{-1} \subseteq (2md)^{-1}$)

Lemma 5

~~Now we show that~~ the sum in (*) is direct.

Proof

For that it is enough to show that the summands are pairwise orthogonal. Let $I := (d) \in \mathcal{P}$ and $J := (c) \in \mathcal{P}$. Suppose $I^2 \mid m\partial$ and $J^2 \mid m\partial$ with $I \neq J$.

Let $I' := (d') = \frac{I}{\gcd(I,J)}$. Then $I' \neq \mathcal{O}$, (otherwise exchange the role of I & J).

Now we have ~~want to show~~ the following ~~equality~~ identity:

(**) $$Tr_m^{I}(\vartheta \mid U_c) = (Tr^{I'}_{m/J^2}\vartheta) \mid U_c \quad \forall \vartheta \in Thm_{/c^2}.$$

(Note that $I'^2 = \frac{I^2}{\gcd(I,J)^2} \mid \frac{m\partial}{J^2}$).

~~If we can show that (**) holds, then~~

The proof of (**) follows by a straightforward (though quite lengthy) calculation. Then

$$Tr_m^{I}(Thm^{new}_{/c^2} \mid U_c) = (Tr^{I'}_{m/J^2}(Thm^{new}_{m/c^2})) \mid U_c. \quad (***)$$

$Thm^{new}_{/c^2}$ is orthogonal to the spaces $Thm_{/c^2/y^2} \mid U_y$, (and $(y)^2 \mid \frac{m\partial}{J^2}$)

But we have chosen I' s.t. $I' \neq \mathcal{O}$ and $I'^2 = \frac{I^2}{\gcd(I,J)^2} \mid \frac{m\partial}{J^2}$

Hence, $Thm^{new}_{/c^2}$ is orthogonal to the space $Thm_{/c^2/d'^2} \mid U_{d'}$. (recall $(d') = I'$). So, from the prev. lemma we obtain

$$Tr_m^{I}(Thm^{new}_{/c^2} \mid U_c) = 0, \text{ from } (***)$$

Since

But often from the prev. lemma and the above fact we have that

$Th^{new}_{m/c^2}|U_c$ is orthogonal to the space $Th_{m/d^2}|U_d$.

Also, in particular the space $Th^{new}_{m/c^2}|U_c$ is orth. to the space $Th^{new}_{m/d^2}|U_d$. Since I & J are arbitrarily chosen, all summands in ⊕ are pairwise orthogonal. Hence the decomposition holds true. $\boxed{}$

~~Proof of [illegible] (4.2)~~

Now we prove ⊕⊕ :

claim: $Tr_m^{I}(\vartheta|U_c) = (Tr^{I'}_{m/J^2}\vartheta)|U_c \quad \forall \vartheta \in Th_{m/c^2}$

From the defn. of the operator Tr_m^{I} we have,

$$\left(Tr^{I'}_{m/J^2}\vartheta\right)|U_c = N(I')^{-2}\sum_{y\in(I'^{-1}/\mathcal{O}\times I'^{-1}/\mathcal{O})} \vartheta\,|_{\frac{1}{2},\frac{m}{c^2}}[y]|U_c.$$

On the other hand we have

$$Tr_m^{I}(\vartheta|U_c) = N(I)^{-2}\sum_{y'\in I^{-1}/\mathcal{O}\times I^{-1}/\mathcal{O}} (\vartheta|U_c)|_{\frac{1}{2},m}[y']$$

$$= N(I)^{-2}\sum_{y'\in I^{-1}/\mathcal{O}\times I^{-1}/\mathcal{O}} \vartheta\,|_{\frac{1}{2},\frac{m}{c^2}}[cy']|U_c \,.\ \circledast_1$$

$\circledast_1$ $\vartheta|U_c\,|_{\frac{1}{2},m}[\lambda,\mu](\tau,z)$

$= e^m[\lambda^2\tau+2\lambda z+\lambda\mu]\,\vartheta|U_c(\tau, z+\lambda\tau+\mu)$

$= e^m[\lambda^2\tau+2\lambda z+\mu\lambda]\,\vartheta(\tau, d(z+\lambda\tau+\mu))$

$\vartheta|_{\frac{1}{2},\frac{m}{c^2}}[c\lambda,c\mu]|U_c(\tau,z) = \vartheta|_{\frac{1}{2},\frac{m}{c^2}}[c\lambda,c\mu](\tau,dz) = e^{\frac{m}{d^2}}[(d\lambda)^2\tau+2d^2\lambda z+d^2\lambda\mu]$

$\cdot$ ~~$\vartheta(\tau,d(z+$~~ $\vartheta(\tau, dz+d\lambda\tau+d\mu)$

Now, since any $y' \in I^{-1}/\mathcal{O} \times I^{-1}/\mathcal{O}$ can be written as $y' = (\lambda_1+\mu_1, \lambda_2+\mu_2)$, where $\lambda_1, \lambda_2 \in I'^{-1}/\mathcal{O}$ and $\mu_1, \mu_2 \in \gcd(I,J)^{-1}/\mathcal{O}$, we have

$$\mathrm{Tr}_m^I(\vartheta|U_C) = N(I)^{-2} \sum_{\lambda_1,\lambda_2 \in I^{-1}/\mathcal{O}} \; \sum_{\mu_1,\mu_2 \in \gcd(I,J)^{-1}/\mathcal{O}} \vartheta|_{\frac{1}{2},\frac{m}{c^2}}[c(\lambda_1+\mu_1), c(\lambda_2+\mu_2)]|_{U_C}$$

$$= N(I)^{-2} \sum_{\lambda_1,\lambda_2} \; \sum_{\mu_1,\mu_2} \vartheta|_{\frac{1}{2},\frac{m}{c^2}}[c\lambda_1, c\lambda_2]|_{\frac{1}{2},\frac{m}{c^2}}[c\mu_1, c\mu_2]|_{U_C}$$

uses the prop. of $|_{\frac{1}{2},\frac{m}{c^2}}$ action and multip. in Heis.

Let us denote

$$\vartheta_1 := \vartheta|_{\frac{1}{2},\frac{m}{c^2}}[c\lambda_1, c\lambda_2].$$

We claim now that

$$\vartheta_1|_{\frac{1}{2},\frac{m}{c^2}}[c\mu_1, c\mu_2] = \vartheta_1. \qquad (5)$$

Enough to ~~We~~ prove the result for a basis element $\vartheta_{\frac{m}{c^2},x}$ of Th_{m/c^2}.

We have from Lemma 1:

$$\vartheta_{\frac{m}{c^2},x}|_{\frac{1}{2},\frac{m}{c^2}}[c\mu_1,\mu_2] = e\left[c\mu_2\left(x+\frac{m}{c^2}c\mu_1\right)\right]\vartheta_{\frac{m}{c^2},x+2mc\mu_1}.$$

Since $\mu_1 \in \gcd(I,J)^{-1} \Rightarrow \gcd(I,J)^{-1} | \mu_1$. Since, $\frac{1}{J} \,\Big|\, \frac{1}{\gcd(I,J)} \,\Big|\, \mu_1$ we have $\mu_1 c \in \mathcal{O}$. So, $2mc\mu_1 \in 2m\mathcal{O}$.

Also, $tr\left(c\mu_2\left(x+\frac{m}{c^2}c\mu_1\right)\right) = tr(c\mu_2 x + m\mu_1\mu_2) = tr(c\mu_2 x) + tr(m\mu_1\mu_2) \equiv 0 \; \mathbb{Z}$

since $c\mu_2 x \in \partial^{-1}$ and $m\mu_1\mu_2 \in \partial^{-1}$.
$(\frac{m_2}{I^2}$ is integral$)$

Hence $v_{\frac{m}{c^2}} \mid x \mid_{\frac12, \frac{m}{c^2}} [c\mu_1, c\mu_2] = v_{\frac{m}{c^2}}, x$.

$\therefore$ (5) follows.

Now we have

$$Tr_m{}^2(v \mid u_c) = N(I)^{-2} \sum_{\lambda_1,\lambda_2 \in I^{-1}/\mathcal{O}} v \mid_{\frac12,\frac{m}{c^2}} [c\lambda_1, c\lambda_2] \mid u_c \sum_{\mu_1,\mu_2} 1$$

$$= N(I)^{-2} N(\gcd(I,J)^2)^2 \sum_{\lambda_1,\lambda_2} v \mid_{\frac12,\frac{m}{c^2}} [c\lambda_1, c\lambda_2] \mid u_c$$

$$= N(I')^{-2} \sum_{\lambda_1,\lambda_2 \in I'^{-1}/\mathcal{O}} v \mid_{\frac12,\frac{m}{c^2}} [c\lambda_1, c\lambda_2] \mid u_c$$

since $(I', J) = 1$

$c\lambda_1$ and $c\lambda_2$ also ranges over all elements of $I'^{-1}/\mathcal{O}$.

Hence,

$$Tr_m{}^2(v \mid u_c) = Tr_{m/J^2}^{I'}(v) \mid u_c \quad \square \quad (9a)$$

~~Main Lemma: The spaces Th_m^{new} is $\mathcal{O}$-invariant~~

Thm proof:

From a ~~well-known result in~~ standard repr. theory (Lemma), we have

$$Th_{m/d^2}^{new} = \bigoplus_{\chi \in \widehat{\mathcal{O}_{\frac{m}{d^2}}}} Th_{m/d^2}^{new,\chi}$$

~~Then by the prev thm.~~ Lineraly this into the decompostn we obtain the decomposit into $M_p(2,c)$ included:

$$Th_m = \bigoplus_{\substack{I=(d) \in P \\ I^2 \mid m\gamma}} \bigoplus_{\chi \in \widehat{\mathcal{O}_{\frac{m}{d^2}}}} Th_{m/d^2}^{new,\chi} \mid u_d$$

// The irreducibility of the parts follows from the following lemma:

Lemma 6 ~~The~~ $\mathrm{Th}_m \to \mathrm{Th}_{m/d^2}$ is a $Mp(2,\mathcal{O})$-module homomorphism. (9a)

Lemma 7 The decomposition (*) is a decomposition into $Mp(2,\mathcal{O})$-submodules.

Proof straight from the def. □

proof By lemma 6 $\mathrm{Th}_m^{old} = \sum_{d\neq 0} \mathrm{Th}_{m/d^2}|U_d$ is $Mp(2,\mathcal{O})$-invariant. Since $Mp(2,\mathcal{O})$ acts unitarily (i.e. $\langle \alpha\vartheta|\alpha, \tilde\vartheta|\alpha\rangle = \langle\vartheta|\tilde\vartheta\rangle$) we see that Th_m^{new} is also $Mp(2,\mathcal{O})$-inv. Hence all $\mathrm{Th}^{new}_{m/d^2}|U_d$ in (*) are $Mp(2,\mathcal{O})$-invariant. □

Recall $\mathcal{O}_m = \{\varepsilon_m \in (\mathcal{O}/2m\mathcal{O})^* \mid \varepsilon_m^2 \equiv 1 \bmod \mathfrak{d}\}$.

The group $\mathcal{O}_m$ acts on Th_m:

$$(\varepsilon, \vartheta_{m,x}) \mapsto \vartheta_{m,\varepsilon x}.$$

Lemma 9 Th_m^{new} is $\mathcal{O}_m$-invt. (To decompose Th_m^{new} into these subspaces)

(follows ~~from~~ a similar consideration as in the proof of Lem. 7).

Lemma 8 The ~~action~~ action of $\mathcal{O}_m$ on Th_m commutes with the action of $Mp(2,\mathcal{O})$.

proof Suffices to check for generators $T^{\alpha} = \left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right)$ $(b \in \mathcal{O})$, $S^* = \left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \sqrt{N\tau}\right)$.

$$\varepsilon(\vartheta_{m,x})|T^* = \vartheta_{m,\varepsilon x}|T^* = \vartheta_{m,\varepsilon x}\, e\!\left(\frac{(\varepsilon x)^2}{4m} b\right)$$

$$\varepsilon(\vartheta_{m,x}|T^{\alpha}) = \varepsilon\, \vartheta_{m,x}\, e\!\left(\frac{b x^2}{4m}\right) = \vartheta_{m,\varepsilon x}\, e\!\left(\frac{b x^2}{4m}\right)$$

RHS's are equal since $\varepsilon^2 \equiv 1 \bmod \mathfrak{d}$.

Similarly for S^* (we skip the proof). □

~~Corollary~~ 8 $\mathrm{Th}_m^{new,\chi}$ ~~and~~ are $Mp(2,\mathcal{O})$-modules.

$v \in \mathrm{Th}_m^{new,\chi}$, $\varepsilon \cdot v$: $\varepsilon(v|\alpha) = (\varepsilon v)|\alpha = \chi(\varepsilon)v|\alpha = \chi(\varepsilon)v|\alpha$

(10) ≤5 (5)

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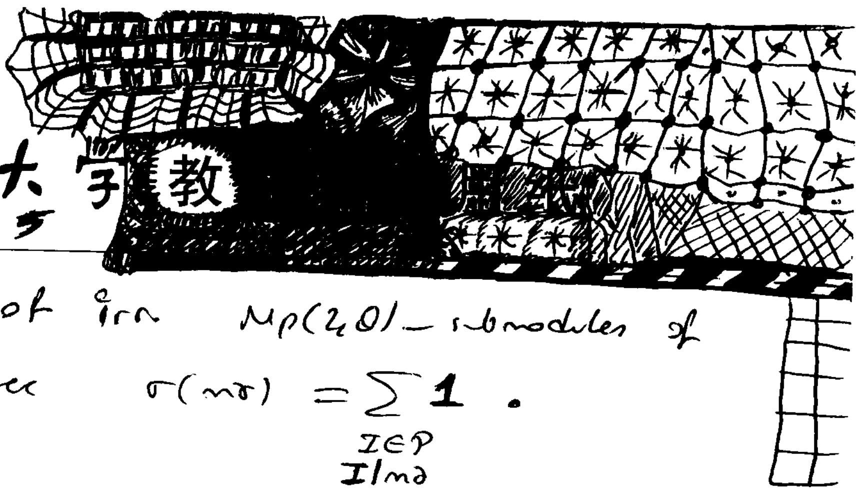

Lemma: The number of irr $Mp(2,\mathbb{Q})$ – submodules of Th_m $\leq \sigma(m\partial)$, where $\sigma(m\partial) = \sum_{\substack{I \in P \\ I \mid m\partial}} 1$.

To show that all are irreducible we need to know that none of them are 0. (for that we need to determine the dim. formula for those subspaces) ~~and~~ ~~Also~~ But the, ~~we~~ after counting the ~~terms~~ summands, we found that there are $\sigma(m\partial)$ many summands. So, if none of them are zero all of them must be irreducible. We will see later dim. formula, which confirms that $Th_m^{\mu,\chi} \neq 0 \ \forall \mu,\chi$.