$T_n f = g$

$T_n f = T_n(f(z)) = \sum f|_M$

$$= \sum_{\substack{a,b,d \\ ad = n \\ d > 0 \\ b \bmod d}} \sum_n k_n e^{2\pi i n z}$$

Jacobi Forms over Number Fields

30/11/

on $SL_2(\mathbb{Z})$

$\phi(\tau, z)$, $\tau \in \mathfrak{h}$.

$\mathfrak{h} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, $z \in \mathbb{C}$.

s.t.

(i) for fix z, $\tau \longmapsto \phi(\tau, \cdot)$ an E.M.F. on $SL_2(\mathbb{Z})$

(ii) For fix τ,
$z \longrightarrow \phi(\cdot, z)$ $\phi(\ )$ theta function over elliptic curve $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$

If instead of elliptic curves, we consider n-dim. ~~Additive~~ abelian varieties {for $k \neq \mathbb{Q}$} whose automorphism ring is contains the ring of integers of a totally real number field say k, with $[k:\mathbb{Q}] = n$ we obtain the analogs of Jacobi forms.
We call it as Jacobi forms over k. They are functions on $\tau \in \mathfrak{h}^n$, $z \in \mathbb{C}^n$ s.t.

(i) fix z,
$\tau \longrightarrow \phi(\tau, z)$ becomes a Hilbert M.F. in $SL(2, \mathcal{O})$
$\mathcal{O}$: ring of integers of K

) fix τ,

$z \longmapsto \phi(\tau, z)$ becomes a Θ function on the Abelian variety $\mathbb{C}^n/\lambda(\tau)$ where $\lambda(\tau)$ consists of vectors

$\sigma_1(\alpha)\tau_1 + \sigma_1(\beta), \ldots \sigma_n(\alpha)\tau_n + \sigma_n(\beta)$

$\alpha, \beta \in \mathcal{O}$, $\sigma_i : k \longrightarrow \mathbb{C}$ $\forall i$

$\tau = (\tau_1, \ldots \tau_n)$

Let k be a totally real number field of degree n. We use $\sigma : k \hookrightarrow \mathbb{R}$, $\forall i$, we use $\mathcal{O}$, ∂ for the ring of integer $\mathcal{O}$ of k and ∂ for the different of k.

$\partial^{-1} = \{ x \in k \mid tr(xy) \subseteq \mathbb{Z}, \forall y \in \mathcal{O} \}$

$\mathfrak{h}$: upper half plane

$\mathbb{C}^n$ = ring with ~~compon~~ component-wise $+$, $\times$

$k \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$(a, z) \longmapsto (\sigma_1(a) z, \ldots \sigma_n(a) z_n)$

So, $\mathbb{C}^n$ becomes a k-algebra with this scalar multiplication we can identify k with its image in $\mathbb{C}^n$ s.t. $a \mapsto (a, 1) \longrightarrow (\sigma_1(a), \ldots \sigma_n(a))$

$tr(z) = \sum_{i=1}^{n} z_i$, $N(z) = \prod_{i=1}^{n} z_i$

$tr(az) = \sum_{i=1}^{n} \sigma_i(a) z_i$

$$\frac{az+b}{cz+d} = \left(\frac{\sigma_1(a)z+\sigma_1(b)}{\sigma_1(c)z_1+\sigma_1(d)}, \dots \frac{\sigma_n(a)z_n+\sigma_n(b)}{\sigma_n(c)z_n+\sigma_n(d)}\right)$$

$$N(cz+d) = \prod_{i=1}^{n} (\sigma_i(c)z_i+\sigma_i(d))$$

The action of $SL(2,K)$ on h^n is,

$$(A,\tau) \longmapsto A\tau := \frac{az+b}{cz+d}$$

The action of $SL(2,k)$ on $h^n \times \mathbb{C}^n$

$$(A,(\tau,z)) \longmapsto A(\tau,z) := \left(A\tau, \frac{z}{cz+d}\right)$$

we write $H(k) = (K \times K) \cdot K$

The multiplication is given by:

$$(x;r)(y,s) = \left(x+y, r+s+\det\begin{pmatrix}x\\y\end{pmatrix}\right)$$

$H(k)$ also acts on $h^n \times \mathbb{C}^n$

$$\underset{(\lambda,\mu)}{\underset{\|}{(x,r)}}(\tau,z) \longrightarrow (x,r)(\tau,z) = (\tau, z+\lambda\tau+\mu)$$

Let $J(k) := SL(2,k) \ltimes H(k)$

The multiplication in $J(k)$ is;

$$(A,h)(B,h') = (AB, \underbrace{(hB)}\cdot h')$$

$S(2,k)$ acts on $H(k)$ via

$$(x,r)A := (xA, r)$$

If we combine the two actions ($SL(2,k)$ and $H(k)$) on $h^n \times \mathbb{C}^n$ we have an action of $J(k)$ on $h^n \times \mathbb{C}^n$ via; $(A,h) = A(h(z,z))$

For fixed $k \in \mathbb{Z}$, $m \in \mathbb{Z}$ (later we will assume $m >> 0$) we have a natural action of $SL(2,k)$ on $\mathrm{Hol}(h^n)$. $(f, A) \longmapsto f|_k A(\tau) = f(A\tau)\, N(c\tau+d)^{-k}$ (4)

We also have an action ~~Π~~ of $SL(2,k)$ on $\mathrm{Hol}(h^n \times \mathbb{C}^n)$: $(f, A) \longmapsto f|_{k,m} A(\tau, z) := f(A(\tau, z))\, N(cz+d)^{-k}$

Here $e^m[*] = \exp(2\pi i\, tr(m\, *))$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$H(k)$ also acts on $\mathrm{Hol}(h^n, \mathbb{C}^n)$

$$\underset{\lambda, \mu}{\overset{}{(f, \underset{\shortparallel}{(x, r)})}} \longmapsto f'|_{k,m}(x, r)(\tau, z) :=$$

$$f(\tau, z + \lambda\tau + \mu)\, e^m[\lambda^2 \tau + 2\lambda z + \lambda\mu + r]$$

$J(k)$ acts on $\mathrm{Hol}(h^n \times \mathbb{C}^n)$;

$$f, (A, h)) \longmapsto f|_{k,m}(A, h) := f|_{k,m} A|_{k,m} h.$$

Defn A Jacobi form of wt. $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}$ with characters $\chi : SL(2, \mathcal{O}) \to S'$ where ke χ has finite index in $S(\mathbb{Z}, \mathcal{O})$, is a function $\phi : h^n \times \mathbb{C}^n \xrightarrow{hol} \mathbb{C}$ s.t.

$$\phi|_{k,m}\, g = \chi(g)\, \phi \quad \forall g \in J(\mathcal{O})$$

$$\left(\text{Here } \chi(\underset{(A,h)}{\overset{\shortparallel}{g}}) = \chi(A) \right)$$

$\boxed{\begin{array}{l} J^k_{k,m}(\chi) \\ J^k_{k,m} = J^k_{km}(1) \end{array}}$ denotes the $\mathbb{C}$-vector space of these functions.

$$J(\mathcal{O}) = SL(2, \mathcal{O}) \ltimes \underset{(\mathcal{O} \times \mathcal{O}) \cdot \mathcal{O}}{\overset{\shortparallel}{H(\mathcal{O})}}$$

$\mathcal{O}$ is a ring of integers of k.

Note that this transformation law implies that we have periodicity w.r.t. τ & z more specifically

(5)

In the z-variable:

$$\phi|_{k,m}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix},(0,\lambda,0)\right)(\tau,z)$$

$$=\phi(\tau,z+\lambda),\ \lambda\in\mathcal{O}$$

$$=\phi(\tau,z)\,\chi\begin{pmatrix}1&0\\0&1\end{pmatrix}=\phi(\tau,z)$$

so $\forall\lambda\in\mathcal{O}$, $\phi(\tau,z+\lambda)=\phi(\tau,z)$

In the τ-variable:

$$\phi|_{k,m}\left(\begin{pmatrix}1&b\\0&1\end{pmatrix},(0,0,0)\right)(\tau,z)=\phi(\tau+b,z)$$

$$=\phi(\tau,z)\,\chi\begin{pmatrix}1&b\\0&1\end{pmatrix}$$

For $b\in\mathcal{U}:=\left\{b\in\mathcal{O}\ \middle|\ \chi\begin{pmatrix}1&b\\0&1\end{pmatrix}=1\right\}$

Then

$$\phi(\tau+b,z)=\phi(\tau,z)\quad\forall b\in\mathcal{U}.$$

We have Fourier expansion of ϕ:

$$\phi(\tau,z)=\sum_{\substack{t\in\mathcal{U}^*\\ r\in\partial^{-1}}}c_\phi(t,r)\,q^t\,\zeta^r\qquad\left(\begin{aligned}q^t(\tau)&=e(t\tau)\\ \zeta^r(z)&=e(zr)\end{aligned}\right)$$

$$\mathcal{U}^*=\{\lambda\in k\ !\ \mathrm{tr}(\mathcal{U}\lambda)\in\mathbb{Z}\}$$

<u>Thm</u> For $k\neq 0$ unless $4mt-r^2>>0$

(Koecher principle) $c_\phi(t,r)=0$ ... $4mt$ ... $r^2)>0$ $\forall$ ...

Ex Let $k = \mathbb{Q}(\sqrt{17})$

$$\vartheta^k(\tau, z) = \sum_{r \in \mathcal{O}} \left(\frac{-14}{N(r)}\right) q^{r^2/8\varepsilon\sqrt{17}} \, \zeta^{\frac{\sigma}{\sqrt{17}}} \qquad (6)$$

$\varepsilon = 4 + \sqrt{17}$, $\varepsilon > 1$

$k = \frac{1}{2}$, $m = \frac{2\varepsilon}{\sqrt{17}}$

$(\vartheta^k)^2 \in J_{1, \frac{\varepsilon}{\sqrt{17}}}(\chi)$

(For quadratic no. fields k, $\mathcal{O}^{\times} = \varepsilon^{\mathbb{Z}} + \{\pm 1\}$ & generators are $\varepsilon, \varepsilon^{-1}, -\varepsilon, -\varepsilon^{-1}$

Remark

If ϕ satisfies a stronger condition, namely $C_\phi(t, r) = 0$ unless $\lim(-\sigma^2) >> 0$.

Then ϕ is called a cusp form.

If $C_\phi(t, r) \neq 0$ only when $\lim(-\sigma^2) = 0$ the ϕ is singular.

We assume $m >> 0$. otherwise it is very likely $\dim J_{k,m} = 0$.

8a

Namely, the index m has a geometric meaning.

For $K = \mathbb{Q}$

For fixed τ, $\phi(\tau, z)$ becomes a theta function on the elliptic curve $\mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$. Such a function has finitely many zeros on $\mathbb{C}/_{\mathbb{Z}+\tau\mathbb{Z}}$. If we count these zeros it turns out to be $2m$. Hence, $\phi \neq 0$, we must have $m \geq 0$. ($m=0$ degenerate case, $\phi(\tau,z)$ const in z).

For an arbitrary number field, for fixed τ, $\phi(\tau, z)$ becomes a theta function on the Abelian variety $\mathbb{C}^n/_{\Lambda(\tau)}$. In this case m becomes the

Chern class of the associated line bundle. If $\phi \neq 0$, the line bundle must be positive, which probably ~~means~~ is equiv. saying that $m >> 0$. Details to be published elsewhere. Not yet worked out completely.

$\left(\phi(\tau, \cdot) = \text{thet. fct. a } \mathbb{C}/_{\mathbb{Z}\tau+\mathbb{Z}} \right)$