§. Dimension Formula:

Recall: we have a decomposition of Th_m into ~~irreducible~~ $Mp(2,\mathcal{O})$-submodules of Th_m:

$$Th_m = \bigoplus_{\substack{I=(d)\in P \\ I^2 | m\partial}} \bigoplus_{\chi \in \hat{\mathcal{O}}_{\frac{m}{d^2}}} Th_{\frac{m}{d^2}}^{new,\chi} | U_d \, . \quad (*)$$

It still remains to show that they are irreducible.
From now on we ~~still~~ restrict to the case of class number 1 ~~case~~. (most of the results hold for also class number > 1; for simplicity we restrict to $Cl(K)=1$)
* for class # > 1 case we don't know they are all results I'm

~~Theorem~~ Lemma: There are exactly $\sigma(m\partial) := \sum_{\substack{a | m\partial \\ a\in P}} 1$ $Mp(2,\mathcal{O})$-submodules of Th_m in the decomposition $(*)$.

Proof: elementary number theory.

Theorem: The number of irreducible $Mp(2,\mathcal{O})$-modules are $\leq \sigma(m\partial)$.

Pf: It uses the theory of Weil representations that we can not represent now.

Consequence: The $Th_m^{new,\chi}$ are irreducible if none of the spaces is zero.
→ but from the dimension formula which we will ~~represent~~ compute now, we see that none of the spaces is 0.
Hence, the decomposition in $(*)$ is a decomposition into irreducible $Mp(2,\mathcal{O})$-submodules.

Theorem: Let $[K:\mathbb{Q}] = n$. We have

$$\dim Th_m^{new,\chi} = 2^n N(m\partial) \prod_{p | m\partial} \frac{1}{2}\left(1 + \frac{M_f(p)}{N(p)}\right) \prod_{p^2 | m\partial} \frac{1}{2}\left(1 - \frac{1}{N(p^2)}\right).$$

Here $\mu_f(\mathfrak{p}) := \mu(\gcd(f,\mathfrak{p}))$, where f is the squarefree divisor of $m\mathfrak{d}$ s.t. $\chi(\varepsilon) = \chi_f\left(\frac{\varepsilon+1}{2}\right)$

Note that μ is the Möbius function of K

(i.e: $\underset{\text{int. ideal}}{\mu(I)} = \begin{cases} (-1)^{\#\mathfrak{p}\mid I} & \text{if } I \text{ is squarefree} \\ 0 & \text{otherwise} \end{cases}$)

Proof: $\dim \mathrm{Th}_m^{\mathrm{new},\chi}$: multiplicity of χ in the character of the $\mathcal{O}_m$-module $\mathrm{Th}_m^{\mathrm{new}}$.

(in a decomposition into irreducible submodules we count the submodules whose character is χ, this number is called the multiplicity of χ).

By orthogonality relations we find hence,

(1) $$\dim \mathrm{Th}_m^{\mathrm{new},\chi} = \frac{1}{|\mathcal{O}_m|} \sum_{\varepsilon\in\mathcal{O}_m} \overline{\chi(\varepsilon)}\, \mathrm{tr}(\varepsilon, \mathrm{Th}_m^{\mathrm{new}}).$$

Hence we have to calculate $\mathrm{tr}(\varepsilon, \mathrm{Th}_m^{\mathrm{new}})$.

Claim: (1) $$\mathrm{tr}(\varepsilon, \mathrm{Th}_m^{\mathrm{new}}) = \sum_{\substack{I=(d)\in\mathcal{P} \\ I^2\mid m\mathfrak{d}}} \mu(I)\, \mathrm{tr}(\varepsilon, \mathrm{Th}_{m/d^2}).$$

We have $$\mathrm{tr}(\varepsilon, \mathrm{Th}_m) = \sum_{\substack{I=(d)\in\mathcal{P} \\ I^2\mid m\mathfrak{d}}} \mathrm{tr}(\varepsilon, \mathrm{Th}_{m/d^2}^{\mathrm{new}}).$$ (**)

Note $\mathcal{O}_m$ also acts on Th_{m/d^2} via:

$(\varepsilon, \vartheta) \longmapsto \varepsilon\vartheta := \mathrm{red}(\varepsilon).\vartheta$, where $\mathrm{red}: \mathcal{O}_m \to \mathcal{O}_{m/d^2}$, $a + 2m\mathfrak{d} \mapsto a + \frac{2m\mathfrak{d}}{d^2}$

$\varepsilon \in \mathcal{O}_m$, $\vartheta \in \mathrm{Th}_{m/d^2}$

So, $\mathcal{O}_m$ also leaves the space $Thm_{/d^2}$ invariant.

So, using the first decomposition of Thm (yesterday), we ~~get~~ can get

$$tr(\varepsilon, Thm) = \sum_{\substack{I=(d) \\ I^2 | m\partial}} tr(\varepsilon, Thm^{new}_{/d^2} | U_d).$$

But from a lemma yesterday which says U_d is an $\mathcal{O}_m$-module homomorphism we get

$$tr(\varepsilon, Thm^{new}_{/d^2} | U_d) = tr(\varepsilon, Thm^{new}_{/d^2}),$$

Hence, ⊛⊛ follows.

Now we prove the claim.

Consider the right hand side of ~~the claim~~ (2). ~~and~~. we put ⊛⊛ into it.

$$\sum_{\substack{I=(d)\in P \\ I^2 | m\partial}} \mu(I) \sum_{\substack{I'=(d')\in P \\ I'^2 | \frac{m\partial}{d^2}}} tr(\varepsilon, Thm^{new}_{/dd'^2})$$

$$= \sum_{\substack{I''=(d'')\in P \\ I''^2 | m\partial}} tr(\varepsilon, Thm^{new}_{/d''^2}) \underbrace{\sum_{\substack{I\in P \\ I | I''}} \mu(I)}_{I''=1} = tr(\varepsilon, Thm^{new}).$$

Now we put (2) into (1) and get

$$\dim Thm^{new,\chi} = \frac{1}{|\mathcal{O}_m|} \sum_{\varepsilon \in \mathcal{O}_m} \overline{\chi(\varepsilon)} \sum_{\substack{I=(d)\in P \\ I^2 | m\partial}} \mu(I)\, tr(\varepsilon, Thm_{/d^2})$$

$$= \frac{1}{|\mathcal{O}_m|} \sum_{\substack{I=(d)\in P \\ I^2 | m\partial}} \mu(I) \sum_{\varepsilon \in \mathcal{O}_m} \overline{\chi(\varepsilon)}\, N\left(\gcd\left(\varepsilon-1, \frac{2m\partial}{d^2}\right)\right)$$

Indeed, ~~Now we~~ when compute $tr(\varepsilon, Thm_{/d^2})$.

we find $= \{ x \in \mathfrak{d}^{-1}/\frac{2m}{d^2} \mid (\varepsilon - 1)x \equiv 0 \mod \frac{2m}{d^2} \}$

$$tr(\varepsilon, Thm_{/d^2}) = N(gcd(\varepsilon - 1, \frac{2m\mathfrak{d}}{d^2})).$$

So, ~~after arranging the terms and using the multiplicativity of the μ function we obtain the~~ result. We leave as an exercise to ~~Now we want to~~ write the double sum in the claimed $\prod$-form.

~~Now for simplicity we assume $m\mathfrak{d}$ is squarefree~~

So all spaces occuring in the decomposition of Thm are nonzero and hence Thm. The fans in (*) are irreducible $Mp(2,\mathcal{O})$-modules.

For finding the 1-dimensional pieces assume for simplicity that $m\mathfrak{d}$ is squarefree.

In this case we have $Thm^{new} = Thm$ and dimension formula becomes

$$~~\dim Thm~~$$

$$\dim Thm^{\chi} = \frac{2^n}{2^s} \prod_{\substack{p \mid m\mathfrak{d} \\ p \text{ even}}} (N(p) + \mu_\chi(p)) \prod_{\substack{p \mid m\mathfrak{d} \\ p \text{ odd}}} \frac{1}{2}(N(p) + \mu_\chi(p))$$

Here s is the # of even prime divisors of $m\mathfrak{d}$.
Note that all factors occuring in this $\prod$-formula are rational integers.

§ 2. Classification of Singular Jacobi Forms:

Theorem 3 $\dim Thm^{\chi} = 1$

$\Leftrightarrow$ χ is totally odd, 2 splits completely, and $m\mathfrak{d}$ satisfies one of the following Eqs.:

- $m\mathfrak{d} = 2\mathcal{O}$ (ex: $m\mathfrak{d} = 6\mathcal{O}$, 3 splits completely)
- $m\mathfrak{d} = 2\mathcal{O}q_1 \cdots q_t$, $q_i \mid 3$ and is of degree 1.

Explanation of "totally real" (see p. 5a)

Remark 1: We do not yet know whether the Theorem is true without the restriction to the class number 1 case. True ~~for all cases~~.

Remark 2: If $[K:\mathbb{Q}]=2$ then $\mathfrak{d}=(\sqrt{D})$, where D is the discriminant of K. Hence $m\mathfrak{d}=2\mathcal{O}$ implies that $m=\frac{2}{\sqrt{D}}\varepsilon$, $\varepsilon\in\mathcal{O}^*$. Here ~~If~~ $\varepsilon>0$ and $\varepsilon'<0$ since $m\gg0$, i.e. $N(\varepsilon)=-1$. (f.u. of K have norm -1)

Pf: Assume $\dim \mathrm{Thm}^{\cancel{new}}=1$.

Then since each factor occurring in the formula is an integer, we have that

$s=n$ (since $s\le n$) ~~(since $\geq s<n$)~~

$N(\mathfrak{p})=2$, $\mu_f(\mathfrak{p})=-1$ } for $\mathfrak{p}\mid m\mathfrak{d}$ even

and

$N(\mathfrak{p})=3$, $\mu_f(\mathfrak{p})=-1$ } for $\mathfrak{p}\mid m\mathfrak{d}$ odd.

Note that $s=n$ means that 2 splits completely.

Vice versa if these above conditions hold, then $\dim \mathrm{Thm}^{\chi}=1$.

Thm

Let $[K:\mathbb{Q}]=2$, say, $K=\mathbb{Q}(\sqrt{D})$ with D fund. disc.

Theorem: Assume $\dim \mathrm{Thm}^{\cancel{new}}\chi=1$. Let ε = fund. unit >1.

Case 1: $m\mathfrak{d}=2\mathcal{O}$ (~~...~~). Then we have Thm^{χ} is spanned by

$$\vartheta^K_{m\mathfrak{d}}(\tau,z)=\sum_{r\in\mathcal{O}}\left(\frac{-4}{N(r)}\right) q^{\frac{r^2}{8\omega}}\,\zeta^{\frac{r}{\omega}}, \quad \omega=\varepsilon\sqrt{D}$$

with ε the fund. unit s.t. $\varepsilon\gg0$.

For $\mathfrak{p}^t \| m\mathcal{O}$ ($\mathfrak{p}$ prime ideal),

define $\varepsilon_{\mathfrak{p}} \in \mathcal{O}/2m\mathcal{O}$ by

$$\begin{cases} \varepsilon_{\mathfrak{p}} \equiv -1 \mod 2\mathfrak{p}^t \\ \varepsilon_{\mathfrak{p}} \equiv +1 \mod 2m\mathcal{O}/\mathfrak{p}^t \end{cases}$$

Then $\varepsilon_{\mathfrak{p}} \in \mathcal{O}_m$. (i.e. $\varepsilon_{\mathfrak{p}}^2 \equiv 1 \mod 4m\mathcal{O}$)

Now let $\chi \in \widehat{\mathcal{O}_m}$.

Set $\displaystyle f := \prod_{\substack{\mathfrak{p} | m\mathcal{O} \\ \chi(\varepsilon_{\mathfrak{p}}) = -1}} \mathfrak{p}$.

Then it is easily verified that

$$\chi(\varepsilon) = \mu\left(\gcd\left(\tfrac{\varepsilon+1}{2}, f\right)\right) \quad \forall \varepsilon \in \mathcal{O}_m$$

(Indeed, this is true for all $\varepsilon_{\mathfrak{p}}$, and the $\varepsilon_{\mathfrak{p}}$ generate $\mathcal{O}_m$).

χ is called <u>totally odd</u> ~~if $f = m\mathcal{O}$~~,

iff. $\chi(\varepsilon_{\mathfrak{p}}) = -1 \quad \forall \mathfrak{p} | m\mathcal{O}$.

(i.e. $f = m\mathcal{O}$)

Case 2 : $m\partial = 6\mathcal{O}$

~~The fu. $\varepsilon > 1$ has norm -1.~~ Then Thm^{χ} is spanned by

$$\vartheta_{m\partial}^{k}(\tau, z) = \sum_{r \in \mathcal{O}} \left(\frac{12}{Nr}\right) q^{\frac{r^2}{24\omega}} \zeta^{\frac{r}{\omega}}, \quad \omega = \varepsilon\sqrt{D},$$

where again ε = fund unit > 1.

Case 3: $m\partial = 2\pi\mathcal{O}$ with $\pi | 3$, $N(\pi) = -3$. Then Thm^{χ} is spanned by

$$\vartheta_{m\partial}^{k}(\tau, z) = \sum_{r \in \mathcal{O}_K} \left(\frac{-4}{Nr}\right)\left(\frac{r}{\pi}\right) q^{\frac{r^2}{8\pi}} \zeta^{r/\pi}.$$

§ 8. Supplementary Remarks:

Not so obvious to construct explicitely Jacobi forms over $K \neq \mathbb{Q}$. Of course

§8. Supplementary remarks

Not so obvious to construct explicitly Jacobi forms over $K \neq \mathbb{Q}$.

Of course, one can take powers of the singular forms. Moreover, they do not even exist for arbitrary K (= $K/\mathbb{Q}$ with an ... $\neq 0$)

But there is another method: A generalization of "Taylor expansion around $z=0$" explained by Skoruppa for $J^{\mathbb{Q}}_{k,1}$. This will be published in a joint paper. As an example: Here is an application of this method to $K = \mathbb{Q}(\sqrt{5})$

(note for $K=\mathbb{Q}(\sqrt{5})$ there are no ~~not exist~~ singular forms since 2 is inert.) $\left(\left(\frac{2}{5}\right) = -1\right)$

Result $K = \mathbb{Q}(\sqrt{5})$, $\mathcal{O} = \mathcal{O}_K$.

$$M_*^K := \bigoplus_{k\in\mathbb{Z}} M_k(SL(2,\mathcal{O})) \quad \text{(Ring of HMF}/K).$$

$$J_{*,\varepsilon/\sqrt{5}} := \bigoplus_{k\in\mathbb{Z}} J^K_{k,\varepsilon/\sqrt{5}} \quad \left(\varepsilon = \frac{1+\sqrt{5}}{2} \text{ fund.} > 1\right)$$

Note $J_{*,\varepsilon/\sqrt{5}}$ is M_*^K-module. $(f, \phi) \mapsto f\phi$

Thm (Gundlach G3)

$$M_*^K = \mathbb{C}[G_2, G_5, G_6, G_{15}] \Big/ \left(G_{15}^2 - \mathrm{Pol}(G_2, G_5, G_6)\right)$$

where G_k is a certain HMF of weight k, respectively.

[illegible]

<u>Thm</u> (Mayer–Hayashida–Khang ??)
$J^{k}_{x, \varepsilon/\sqrt{5}}$ is a M^{k}_{δ}-module of rank 8. More precisely,

$$J^{k}_{x, \varepsilon/\sqrt{5}} = \bigoplus_{j=1}^{8} F_{\lambda_j}\, \mathbb{C}[G_{2}, G_{5}, G_{6}],$$

where F_{λ_j} $\underbrace{\in J^{k}_{\lambda_j, \varepsilon/\sqrt{5}}}_{\text{is a certain Jacobi form in}}$, and where

$$\lambda_1\ \lambda_2 \ldots \lambda_8 = 2\ 4\ 5\ 6\ 7\ 11\ 14\ 15.$$

$$J^{k}_{k, \varepsilon/\sqrt{5}} \cong (M^{k}_{\delta})^{8} / U$$

where $U \subset (M^{k}_{\delta})^{8}$

$$= \sum_{i=1}^{8} F_{\lambda_i} M^{k}_{\delta} \quad \text{f. index.}$$

λ_i / JF's

$$0 \to (M^{k}_{\delta})^{?} \to (M^{k}_{\delta})^{a_1} \to (M^{k}_{\delta})^{8} \to J^{k}_{k, \varepsilon/\sqrt{5}} \to 0$$

$[2, 4, 5]$ syzygies

$\vartheta^{k} \in J^{k}_{\frac{1}{2}, \frac{2\varepsilon}{\sqrt{17}}}$.

- For $K \neq \mathbb{Q}$ one has in addition $\sigma_J(4mt - r^2) \geq 0$ (Koecher Principal) (proof later) (for $K = \mathbb{Q}$ we have to add ~~assume~~ this condition to the definition of Jacobi forms)
- If ϕ satisfies a stronger condition, namely: $c_\phi(t, r) = 0$ unless $4mt - r^2 \gg 0$, then ϕ is called a cusp form.
- If $c_\phi(t, r) \neq 0$ only for $4mt - r^2 = 0$, then ϕ is called singular. ~~We shall see dim $J_{k,m}$ ...~~

From now on we assume $m \gg 0$. We need this for the so called theta-expansion which we shall explain now and which has various important consequences

Thm: We have the theta expansion for $\phi \in J_{k,m}$:

$$\phi(\tau, z) = \sum_{\rho \in \partial^{-1}/2m\mathcal{O}} h_\rho(\tau)\, \vartheta_{m,\rho}(\tau, z),$$

(e.g. dim $J^{k}_{k,m}$ ... + Koecher principle ... et. Very likely: dim $J^{k}_{k,m} = 0$, unless $m \gg 0$ (open, but Skoruppa and myself have a ...))

where

$$\vartheta_{m,\rho}(\tau, z) = \sum_{\substack{r \in \partial^{-1} \\ r \equiv \rho \bmod 2m\mathcal{O}}} q^{\frac{r^2}{4m}} \zeta^{r}$$

$q^{t}(\tau) = e(t\tau)$

$\zeta^{r}(z) = e(rz)$

(the trace ... this?)

and $h_\rho \in \mathrm{Hol}(\mathfrak{h}^n)$. ~~For the sake of simplicity~~ $\chi = 1$

Proof: Let $D = -4mt + r^2$

Denote $C_\phi(D, r) := c_\phi\left(\frac{r^2 - D}{4m}, r\right)$

Using (b) of definition of Jacobi forms, we write

$$\phi(\tau, z) = \sum_{\substack{\frac{r^2 - D}{4m} \in \mathcal{O}^{\#} \\ r \in \partial^{-1}}} C_\phi(D, r)\, q^{\frac{r^2 - D}{4m}} \zeta^{r} \quad (1)$$

$$(2) \quad = \sum_{r \in \partial^{-1}} q^{\frac{r^2}{4m}} \zeta^{r} \sum_{\frac{r^2 - D}{4m} \in \partial^{-1}} C_\phi(D, r)\, q^{\frac{-D}{4m}}$$

From (a) of the same defn we have

$\phi|_{k,m}([\lambda, \mu]) = \phi$ for $\lambda, \mu \in \mathcal{O}$

$J(\mathcal{O}) = S((2, \mathcal{O}) \ltimes H(\mathcal{O})$

$H(\mathcal{O}) = (\mathcal{O} \times \mathcal{O}) \times \mathcal{O}$

$$\phi|_{k,m}[\lambda,\theta](\tau,z) = \phi(\tau, z+\lambda\tau)\, e(m\lambda^2\tau + 2m\lambda z)$$

$$= \phi(\tau,z)$$

So, (1) becomes

$$(3)\quad \phi(\tau, z+\lambda\tau)\, e(m\lambda^2\tau + 2m\lambda z) = \sum_{\substack{\cancel{\frac{r^2-D}{4m}} \in \partial^{-1} D \\ r\in\partial^{-1}}} C_\phi(D,r)\, q^{\frac{r^2-D}{4m}}\, \zeta^r .$$

note $\zeta^r(z) = e(rz)$

$\zeta^r(z+\lambda\tau) = e(rz + r\lambda\tau)$

But $$\phi(\tau, z+\lambda\tau) = \sum_{\substack{\cancel{\frac{r'^2-D'}{4m}} \in \partial^{-1} D \\ r\in\partial^{-1}}} C_\phi(D,r')\, q^{\frac{r'^2-D}{4m}}\, \zeta^{r'} \cdot e(r'\lambda\tau)$$

$q^t(\tau) = e(t\tau)$

So, $$\phi(\tau,z+\lambda\tau)e(m\lambda^2\tau + 2m\lambda z) = \sum_{\substack{\frac{r'^2-D}{4m} \\ r'\in\partial^{-1}}} C_\phi(D, r')\, q^{\frac{r'^2-D}{4m}}\, \zeta^{r'}\, e((m\lambda^2 + r'\lambda)\tau + 2m\lambda z)$$

$$= \sum_{\substack{\cancel{\frac{r'^2-D'}{4m}\in\partial^{-1}} D \\ r'\in\partial^{-1}}} C_\phi(D,r')\, q^{\frac{r'^2-D}{4m} + m\lambda^2 + r'\lambda} \cdot \zeta^{r'+2m\lambda}$$

$$q^{\frac{(r'+2m\lambda)^2 - D}{4m}}$$

If we do the substitution $r' + 2m\lambda \longmapsto r$. Then we see ~~from~~ using (3) that

$$\underline{C_\phi(D,r) = C_\phi(D, r-2m\lambda)} \qquad \forall \lambda\in\theta .$$

That means $C_\phi(D,r)$ depends only on $r \bmod 2m\theta$.

Hence, (1) becomes

$$\otimes\quad \phi(\tau,z) = \sum_{\rho\in\partial^{-1}/2m\theta} \underbrace{\sum_{\substack{r\in\partial^{-1} \\ r\equiv\rho\ 2m\theta}} q^{\frac{r^2}{4m}}\, \zeta^r}_{\theta_{m,\rho}} \underbrace{\sum_{\cancel{\frac{r^2-D}{4m}}\in\partial^{-1} D} C(D,\rho)\, q^{\frac{-D}{4m}}}_{h_\rho(\tau)}$$

Thus

$$\phi(\tau,z) = \sum_{\rho\in\partial^{-1}/2m\theta} h_\rho(\tau)\, \theta_{m,\rho}(\tau,z).$$