Section 2 : Basic Properties :

Assume for simplicity $X \equiv 1$. (For $X \neq 1$ the following statements hold also true with some technical modifications)

By ~~the~~ (1) : ~~one~~ one has

$$\phi(\vec{\tau}+\alpha; \vec{z}) = \phi(\vec{\tau}, \vec{z}) \quad \forall \alpha \in \mathcal{O}_K \, .$$

$$\phi(\vec{\tau}; \vec{z}+\alpha) = \phi(\vec{\tau}, \vec{z}) \quad \forall \alpha \in \mathcal{O}_K \, ,$$

where $\vec{z}+\alpha = (z_1+\sigma_1(\alpha), \dots, z_n+\sigma_n(\alpha))$

$\vec{\tau}+\alpha = (\tau_1+\sigma_1(\alpha), \dots, \tau_n+\sigma_n(\alpha))$.

(ϕ ~~has~~ is periodic) wrt ~~both~~ each coordinate)

Hence, ϕ has a Fourier expansion of the form:

$$(1') \quad \phi(\tau; z) = \sum_{m r \in \delta_K^{-1}} c_\phi(m r) \exp\Big(2\pi i \big(\sum_{i=1}^{n} \sigma_i(m) \tau_i + \sigma_i(r) z_i\big)\Big).$$

namely:

- ~~Since~~ ~~For~~ Assume ϕ is as above.

~~If~~ $K \neq \mathbb{Q}$, one has in addition. $c_\phi(n, r) = 0$ unless $\sigma_j(4mn - r^2) \geq 0 \; \forall j$ (Koecher principle) (for $K = \mathbb{Q}$ this has to be assumed)

- If ϕ satisfies a stronger condition, namely: $c(n, r) = 0$ unless ~~$c_\phi(n r) \neq 0 \Rightarrow$~~ $4mn - r^2 > 0$ it is called a cusp form. (i.e. $\sigma_j(4mn - r^2) > 0 \; \forall j$)

- If $c_\phi(n, r) \neq 0$ only for $4mn - r^2 = 0$ then ϕ is called singular.

- ~~et~~

Proof of the Köcher principle (see later after ϑ-expansion).

Let $\phi \in J_{k,m}^{K}(X)$. $\partial_K^{-1} = \{x \in K \mid tr(xy) \in \mathbb{Z}\ \forall y \in \mathcal{O}_K\}$

We have ~~the~~ theta expansion of ϕ in the following way: (6)

Thm:

$$\phi(\vec{\tau}, \vec{z}) = \sum_{\substack{\rho \in \partial_K^{-1} / 2m\mathcal{O}_K}} h_\rho(\vec{\tau})\, \vartheta_{m,\rho}(\vec{\tau}, \vec{z}),$$

where

$$\vartheta_{m,\rho}(\vec{\tau}; \vec{z}) = \sum_{\substack{r \in \partial_K^{-1} \\ r \equiv \rho \bmod 2m\mathcal{O}_K}} \exp\left(2\pi i \sum_{i=1}^{n}\left(\sigma_i\left(\frac{r^2}{4m}\right)\tau_i + \sigma_i(r) z_i\right)\right).$$

and $h_\rho \in Hol(\mathfrak{H}^n)$. ~~In fact $h_\rho \in M_{k-\frac{1}{2}}(\Gamma)$.~~

Pf: Let $D = -4mn + r^2$

Let us define

$$C(D, r) := c\left(\frac{r^2 - D}{4m}, r\right) \qquad \left(\text{since } \frac{r^2-D}{4m} = n\right)$$

Now we can write using the Fourier expansion of ϕ:

$$(2) \quad \phi(\vec{\tau}; \vec{z}) = \sum_{\substack{\frac{r^2-D}{4m} \in \partial_K^{-1} \\ D << 0}} C(D, r) \exp\left(2\pi i \sum_{i=1}^{n}\left(\sigma_i\left(\frac{r^2-D}{4m}\right)\tau_i + \sigma_i(r) z_i\right)\right)$$

$$(3) \quad = \sum_{r \in \partial_K^{-1}} \exp\left(2\pi i \sum_{i=1}^{n}\left(\sigma_i\left(\frac{r^2}{4m}\right)\tau_i + \sigma_i(r) z_i\right)\right) \sum_{\substack{D << 0 \\ \frac{r^2-D}{4m} \in \partial_K^{-1}}} C(D, r) \exp\left(2\pi i \sum_{i=1}^{n} \sigma_i\left(\frac{-D}{4m}\right)\tau_i\right)$$

Recall: (1) — $\phi|_{k,m}\, g = \chi(g)\phi \quad \forall g \in SL(\mathcal{O}_K)^J$

$\chi(A, [\lambda, \mu]) = \chi(A) \quad \forall \lambda, \mu \in \mathcal{O}_K,\ A \in SL(2, \mathcal{O}_K)$.

Now using (1) for ~~$g \in$~~ $g = [\lambda, 0]$, $\lambda \in \mathcal{O}_K$, we have

$$\phi|_{k,m}[\lambda, 0](\vec{\tau}; \vec{z}) = \phi(\vec{\tau}; \vec{z})$$

But

$$\phi|_{k,m}[\lambda, 0](\vec{\tau}; \vec{z}) = \phi(\vec{\tau}; \vec{z} + \lambda\vec{\tau}) \exp\left(2\pi i \sum_{i=1}^{n} \sigma_i(\lambda^2 m)\tau_i + \sigma_i(2m\lambda) z_i\right)$$

$$= \phi(\vec{\tau}; \vec{z}) \quad \text{— from (1).}$$

Now we subs. (2) in the above equality and get:

~~But using (1) we get~~ ~~and putting in (2) we get~~

$$\cancel{\sum_{n, r \in \partial_K} c_\phi(n, r) \exp\left(2\pi i \left(\sum_{i=1}^{n} \sigma_i(n)\tau_i + \sigma_i(r) z_i\right)\right)} = \sum_{n', r' \in \partial_K^{-1}} C_\phi(n', r') \exp\left(2\pi i \sum_{i=1}^{n} \sigma_i(n')\tau_i \ldots (z_i + \lambda\tau_i)\right)$$

$$\cancel{\exp\left(-2\pi i \sum_{i=1}^{n} \sigma_i(\lambda^2 m)\tau_i + \sigma_i(2m\lambda) z_i\right)}$$

But ~~this~~ implies that

$$\cancel{\sum_{n,r \in \partial_K^{-1}} c_\phi(n,r) \exp\left(2\pi i\left(\sum_{i=1}^n \sigma_i(n)\tau_i + \sigma_i(r) z_i\right)\right) = \sum_{n',r' \in \partial_K^{-1}} c_\phi(n',r') \exp\left(2\pi i\left(\sum_{i=1}^n \sigma_i(\lambda^2 m + r'\lambda + n')\tau_i + \sigma_i(r' + 2m\lambda) z_i\right)\right)}$$

$$\cancel{= \sum_{n',r} c_\phi(n', r - 2m\lambda) \exp\left(2\pi i\left(\sum \sigma_i(\lambda^2 m + (r - 2m\lambda)\lambda + n')\tau_i + \sigma_i(r) z_i\right)\right)}$$

Hence,

$$\sum_{\substack{\frac{r^2-D}{4m} \in \partial_K^{-1} \\ D << 0}} c(D,r)\, e\left(\sum_{i=1}^n \sigma_i\left(\frac{r^2-D}{4m}\right)\tau_i + \sigma_i(r) z_i\right)$$

$$= \sum_{\frac{r'^2-D}{4m} \in \partial_K^{-1}} c(D,r')\, e\left(\sum_{i=1}^n \sigma_i\left(\frac{r'^2-D}{4m}\right)\tau_i + \sigma_i(r') z_i + \sigma_i(\lambda) \tau_i\right)$$

$$\cdot\ e\left(2\pi i \sum_{i=1}^n \sigma_i(\lambda^2 m)\tau_i + \sigma_i(2m\lambda) z_i\right)$$

So,

$$\sum_{\substack{\frac{r^2-D}{4m} \in \partial_K^{-1} \\ D << 0}} c(D,r)\, e\left(\sum_{i=1}^n \sigma_i\left(\frac{r^2-D}{4m}\right)\tau_i + \sigma_i(r) z_i\right) \quad \text{-- LHS}$$

$$= \sum_{\frac{r'^2-D}{4m} \in \partial_K^{-1}} c(D,r')\, e\left(\sum_{i=1}^n \sigma_i\left(\underbrace{\frac{r'^2-D}{4m} + \lambda r' + \lambda^2 m}_{= \frac{(r'+2m\lambda)^2 - D}{4m}}\right)\tau_i + \sigma_i\left(\underbrace{2m\lambda + r'}_{r}\right) z_i\right)$$

$$\cancel{\text{LHS} = \sum_{\frac{r'^2-D}{4m}} c(D, r - 2m\lambda)\, e\left(\sum_{i=1}^n \sigma_i\right.}$$

~~But for $r' = 2m\lambda - r$ we have~~

$$\cancel{-4m\left(\frac{r'^2 - D}{4m} + \lambda r' + \lambda^2 m\right) + r'^2}$$

$$\cancel{= \frac{-4m r'^2 + 4mD - 4m\lambda r' - 4m^2\lambda^2 + 4mr'^2}{4m} = D - \lambda r - m\lambda^2}$$

But

$$-4m\left(\frac{r'^2-D}{4m}+\lambda r'+\lambda^2 m\right)+(2m\lambda+r')^2$$

$$=\cancel{-r'^2}+D\,\cancel{-4m\lambda r'}\,\cancel{-4m\lambda^2 m}\;\cancel{+4m^2\lambda^2}\;\cancel{+r'^2}\;\cancel{+4m\lambda r'}$$

$$= +D$$

So, if we do the substn $r'+2m\lambda \longmapsto r$ we get

$$C(D,r)=C(D,r+2m\lambda) \quad \text{for } \lambda\in\mathcal{O}_K$$

i.e: $C(D,r)$ depends only on r mod $2m\mathcal{O}_K$.

Hence (B') becomes :

$$\sum_{\rho\in\mathfrak{d}^{-1}/2m\mathcal{O}}\underbrace{\sum_{\substack{r\in\mathfrak{d}_K^{-1}\\ r\equiv\rho \bmod 2m\mathcal{O}_K}} e\Big(\sum_{i=1}^{n}\big(\sigma_i\big(\tfrac{r^2}{4m}\big)\tau_i+\sigma_i(r)z_i\big)\Big)}_{\vartheta_{m,\rho}(\vec{\tau}:\vec{z})}\;\underbrace{\sum_{\substack{D\ll 0\\ \frac{\rho^2-D}{4m}\in\mathfrak{d}_K^{-1}}} C(D,\rho)\,e\Big(\sum_{i=1}^{n}\sigma_i\big(\tfrac{-D}{4m}\big)\tau_i\Big)}_{h_\rho(\tau)}$$

Hence,

$$(3)\quad \varphi(\tau,z)=\sum_{\rho\in\mathfrak{d}^{-1}/2m\mathcal{O}} h_\rho(\tau)\,\vartheta_{m,\rho}(\tau,z)\,. \quad \blacksquare$$

~~Th. The h_ρ are Hilbert modul forms.~~ // For the proof we need to def. $|_k$ for $k\in\frac{1}{2}\mathbb{Z}$. $\Longrightarrow$

Proof Now we have to show h_ρ is an Hilbert modular form.

we have (3) $\chi(g)\,\phi=\phi|_{k,m}\,g$ for $g\in SL(2,\mathcal{O}_K)^J$.

$l:=|\mathfrak{d}^{-1}/2m\mathcal{O}|\;(=N(2m\mathfrak{d}))$

So from (3)

$$\chi(g)\,\phi=\sum_{\rho\in\mathfrak{d}_K^{-1}/2m\mathcal{O}} h_\rho|_{k-\frac{1}{2}}\,g\;\;\vartheta_{m,\rho}|_{\frac{1}{2}}\,g$$

(~~$\operatorname{span}(\vartheta_{m,\rho})$ is an $SL(2,\mathcal{O}_K)$ module~~)

Th. $(\vartheta_{m,\rho})$ $\exists M: SL(2,\mathcal{O})\to GL_l(\mathbb{C})$ s.t.

$$\begin{pmatrix}\vdots\\ \vartheta_{m,\rho}\\ \vdots\end{pmatrix}\Big|_{\frac{1}{2},m} g = M(g)\begin{pmatrix}\vdots\\ \vartheta_{m,\rho}\\ \vdots\end{pmatrix}, \qquad M(g)\in GL_l(\mathbb{C})$$

$$M(g)=\big(M(g)_{\rho\sigma}\big)_{\rho,\sigma\in l}$$

$$\chi(g)\sum h_\rho\,\vartheta_{m,\rho}=\sum_{\rho\in\mathfrak{d}^{-1}/2m\mathcal{O}} h_\rho|_{k-\frac{1}{2}}\,g\sum_\sigma M(g)_{\rho\sigma}\,\vartheta_{m,\sigma}$$

Hence, $\chi(g)\,h_\sigma=\sum_\rho h_\rho|_{k-\frac{1}{2}}\,g\cdot M(g)_{\rho\sigma}$

Th. $M(g)=1$ for $g\in\Gamma(4m\mathfrak{d})^*$ where $\Longrightarrow$

TÜBİTAK

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Konu : Hatice BOYLAN

⇒ $\Gamma(q_n\mathfrak{p})^{*} \subseteq SL(2,\mathcal{O})$, ~~sub that the~~ where proof and the ... part coefficients ~~equal~~ $\Gamma(q_n\mathfrak{p}) = \{A \in SL(2,\mathcal{O}) \mid A \equiv 1 \; q_n\mathfrak{p}\}$, then $\chi(g) = \ldots$ $\forall g \in \Gamma(q_n\mathfrak{p})^{*}$, i.e. ... HMF.

İLGİLİ MAKAMA

TÜRKİYE BİLİMSEL VE TEKNOLOJİK ARAŞTIRMA KURUMU, geleceğin bilim adamlarının yetişmesine katkıda bulunmak üzere çeşitli burs ve destek programları yürütmektedir. Eğitimlerini başarı ile sürdüren ve tamamlayan öğrencilerden, verilen burslar nedeniyle bir karşılık beklenmemektedir.

Bilkent Üniversitesi Matematik Bölümü doktora öğrencisi **Hatice BOYLAN** Mart 2006'dan beri Yurt İçi Doktora Burs Programımız kapsamında desteklenmektedir. Adı geçene halen aylık **1.500,00 YTL** burs ödenmekte olup, öğrencilik durumunun devam etmesi kaydıyla, **31 Ağustos 2010** tarihine kadar ödemeleri devam ettirilecektir.

Bilgilerinize sunarım.

→ For this we need to introduce

$\widetilde{SL}(2,\mathcal{O})$ = metaplectic cover of $SL(2,\mathcal{O})$
$= \{(A,w) \mid w : \mathfrak{H}^{n} \to \mathbb{C}$ hol., $w^2(\tau) = \prod(\sigma_i(c)\tau_i + \sigma_i(d))\}$

Composition law: $(A,w)(B,v) = (AB, w(B\tau)v(\tau))$.

For each ... $\widetilde{SL}(2,\mathcal{O})$ acts on ... $(\mathfrak{H}^{n})$ (on Hol($\mathfrak{H}^n \times \mathbb{C}^{\cdot}$)) via $|_k$ ... where ... is defined by the same formulas as before but with $\prod(\sigma_i(c)\tau_i + \sigma_i(d))^{-k}$ replaced by w^{-2k}

Saygılarımla,

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Bilim İnsanı Destekleme
Daire Başkanı V.

[illegible]

Corollary Let $\phi \in \mathcal{T}^{L}_{n,m}$. Then

$$C_\phi(D,\nu) = 0 \quad \text{unless } \sigma_i(D) \leq 0 \ \forall i$$

(i.e. the "Koecher principle" holds true for ϕ)

~~pf.~~ $\phi = \sum_\rho h_\rho \vartheta_{m,\rho}$.

But $h_\rho = \sum C_\phi(D,\rho)$ ~~$e(\ldots)$~~

Koecher principle for Hilbert modular forms implies $C_\phi(D,\rho) = 0$ unless $\sigma_i(D) \leq 0 \ \forall i$. $\square$

Section 3 – Singular Jacobi forms over K:

Recall: $\phi \in J^K_{k,m}(\mathcal{X})$ is called singular if $c_\phi(t,r) \neq 0$ only if $4mt - r^2 = 0$.

That means iff $c_\phi(D,\rho) \neq 0$ only for $D=0$.

If we look at the theta expansion

$$\phi(\vec{\tau}:\vec{z}) = \sum_{\rho \in \mathfrak{d}_K^{-1}/2m\mathcal{O}_K} h_\rho(\vec{\tau})\, \vartheta_{m,\rho}(\vec{\tau}:\vec{z}) \quad \text{then}$$

ϕ is singular iff h_ρ is a constant for all ρ.

(since iff $h_\rho(\vec{\tau}) = c_\phi(0,\rho)$)

recall $h_\rho = \sum_D c_\phi(D,\rho)\, e(\ldots)$.

~~Hence~~, Note $h_\rho \equiv$ constant iff $\mathrm{wt}(h_\rho) = k - \frac{1}{2} = 0$.

Hence, we have

Thm: Let $\phi \in J^K_{k,m}(\mathcal{X})$, $\phi \neq 0$. TFAE:

i) ϕ is singular

ii) $k = \frac{1}{2}$

iii) $\phi \in \mathrm{span}\{\vartheta_{m,\rho}\}_{\rho \in \mathfrak{d}_K^{-1}/2m\mathcal{O}_K}\} =: Th_m$

(iv) $\phi \in$ ~~span $\{\vartheta_{m,\rho}\}_{\rho \in \mathfrak{d}^{-1}/2m\mathcal{O}}$~~ $Th_m^{\widetilde{SL(2,\mathcal{O})},\mathcal{X}}$

(Notation: If X is a $\widetilde{SL(2,\mathcal{O})}$-module, then $X^{\widetilde{SL(2,\mathcal{O})},\mathcal{X}} := \{\vartheta \in X \mid \vartheta|g = \mathcal{X}(g)\vartheta \ \forall g \in \widetilde{SL(2,\mathcal{O})}\}$)

Hence, to determine all singular Jacobi ~~modular~~ forms we have to determine for each m, the 1-dim. $\widetilde{SL(2,\mathcal{O})}$ submodules of Th_m. ~~SG~~.