* Shangai – Lecture I – 11.08.2009 (1)

0 – Introduction :

Recall : A Jacobi form on $SL(2,\mathbb{Z})$ is a function

$\phi(\tau, z)$, $\tau \in h$, $z \in \mathbb{C}$ $\left(h = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \} \right)$.

such that ϕ : i) for a fixed z, ~~is a modular form~~

τ ~~...~~ $\tau \mapsto \phi(\tau, z)$ (a elliptic & modular form on $SL(2,\mathbb{Z})$)

ii) for fixed τ,

$z \mapsto \phi(\tau, z)$ is a theta function on

the elliptic curve $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ → complex manifold (1 dim, genus 1) compact, group

(theta function on the elliptic curve: holomorphic sections of an elliptic curve. elliptic curve over $\mathbb{C}$ ~~algebraic curve: non-sing. on alg. variety + which is smp. non-sing. ...~~)

Instead of elliptic curves, we study now n-dim. Abelian varieties (n-dim, compact ~~complex manifold~~ ...) (Abelian variety = compact, complex manifold which is a pp and algebraic) whose end. ring is isom. to the ring of integers in a totally real # field of degree n, say K. We want to f. the analogs of Jacobi forms. We call them Jacobi forms over K.

The Jacobi forms over K are functions for

$\tau \in h^n$, $z \in \mathbb{C}^n$

such that,

i) for fixed z

the fn $\tau \mapsto \phi(\tau, z)$ a Hilbert mod form on $SL(2, \mathcal{O})$ ($\mathcal{O}$ max. order in K)

ii) for fixed τ,

$z \mapsto \phi(\tau, z)$ is a theta fn. ~~...~~

(ie : hol. section ~~over~~ in a line bundle over ~~the~~ Abelian ~~Algebraic~~ variety $\mathbb{C}^n / \lambda(\tau)$ where $\lambda(\tau)$ equals

proj. alg. variety: common zeros of finite polynomials in a proj space.
any ~~non-singular~~ alg. variety is a complex manifold.
But any complex manifold is not always bihol. equivalent to ~~complex manifold~~ an alg. variety. (we have when this alg. variety as a complex manifold)

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Reiseinformation

Buchungscode: YK5HN3

- alg. curve: subset of non-sing. points whose non-sing. locus viewed as complex manifold as 1-dim.

~~studying when a complex manifold is an alg. variety one has to study the embed. of this complex manifold into a proj. space. Under this image is an alg. to study studying the embeddings one has to~~ ~~If we have an embedding which as "suff many hol. sections" then we say that this complex manifold is an alg. variety.~~

Wir danken Ihnen für Ihre Buchung und wünschen Ihnen eine gute Reise.
Mit freundlichen Grüßen

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Section 0 – Introduction

Section 1 – Definition of Jac. For. over K, $S\ell(2,0)^n$, a^n, $\mathfrak{H}^n$, $S\ell(2,O)$, action, action of $S\ell(2,O) \ltimes O^{2n}$, Ex J^k on $Hol(\mathfrak{H}^n \times \mathbb{C}^n)(\partial, m \partial^{-1})$, Def. of JF of wt k e.g. ind. m, char. $\chi : S\ell(2,O) \to S^1$, Rat. lat. $\mathfrak{F} \subseteq \frac{1}{2}\mathbb{Z}$ $m \gg 0$

Section 2 – Basic Properties $J^k_{k,m}$
Fourier exp., Koecher principle, ~~theta expansion~~ cusp form, singular form, theta expansion

Section 3 – Singular Jacobi forms ~~over~~ K
Th. $\phi \in J^k_{k,m}$ singular iff $k = \frac{1}{2}$
~~Them~~ to Sil exp.:
$J^k_{\frac{1}{2}, m} = \mathfrak{S}_{p}(\partial_{m,\beta})$ $\widetilde{S\ell(2,O)}$

$S\ell(2,O)$
$\frac{1}{2}$ alg
$S\ell(2,O)$
Sim. as hep
Wl

(Next Day) Section 4 – Explicit description of all singular Jacobi forms over K.

Ref.
$J^k_{k,m}$
$k \in \frac{1}{2}\mathbb{Z}$

set of vectors $\sigma_1(\alpha)\tau_1+\sigma_1(\beta)$, $\sigma_2(\alpha)\tau_2+\sigma_2(\beta)$, $---\ \sigma_n(\alpha)\tau_n+\sigma_n(\beta)$, where

(2)

$\alpha, \beta \in \mathcal{O}$ and $\sigma_i : K \hookrightarrow \mathbb{R}$, $i=1 \dots n$.
where $\mathcal{O}$ is the ring of integers, where $\tau=(\tau_1 \dots \tau_n)$.

We shall keep these notations throughout the talk.
~~There~~ So far there ~~are~~ were no explicit examples of Jacobi forms over # fields (a part from some theta functions)
(But at the end of my talks you will see many examples)

1 - The Definition of Jacobi forms over K:

The first definition of Jacobi forms over K was given by Richter and Bringmann. Richer studied theta functions over # fields. All Jacobi forms over $\mathbb{Q}(\sqrt{5})$ were determined by Skoruppa & Hayashida & Boylan which will appear soon.

The ~~plan~~ aim of my talk:

i) To ~~explain~~ give ~~the general~~ the precise definition and basic properties of Jacobi forms over K and

ii) To classify all singular Jacobi forms over K.
(the notion "singular" will be explained later)

(1)

Let K be a totally real # field of degree n. We use σ_i $(i=1,\dots,n)$ for the embeddings of K into $\mathbb{R}$. We use $\mathcal{O}$, ∂ $(\partial^{-1}=\{x\in K \mid tr(xy)\in\mathbb{Z},\ \forall y\in\mathcal{O}_K\})$ for the ring of integers of K and the different of K. $\mathfrak{h}$: upper half plane

$\mathbb{C}^n$ is a ring via componentwise addition and multiplication.

The ring $\mathbb{C}^n$ becomes a K-algebra via the map

$$K\cdot\mathbb{C}^n\longrightarrow\mathbb{C}^n$$
$$(a,z)\longmapsto(\sigma_1(a)z_1,\dots,\sigma_n(a)z_n)$$

($\mathbb{C}^n$ is a v.s. over K; $\mathbb{C}^n$ is a ring + scalar multiplication with K)

We identify K with its image in $\mathbb{C}^n$ via the map

$$a\longmapsto a\cdot(1,\dots,1)=(\sigma_1(a),\dots,\sigma_n(a))$$

So, K becomes a subfield of $\mathbb{C}^n$.

$$tr(z):=\sum_{i=1}^{n} z_i \qquad N(z)=\prod_{i=1}^{n} z_i$$

So, the expressions below become meaningful:

$$tr(az)=\sum_{i=1}^{n}\sigma_i(a)z_i$$

$$\frac{az+b}{cz+d}=\left(\frac{\sigma_1(a)z_1+\sigma_1(b)}{\sigma_1(c)z_1+\sigma_1(d)},\dots,\frac{\sigma_n(a)z_n+\sigma_n(b)}{\sigma_n(c)z_n+\sigma_n(d)}\right)$$

$$N(cz+d)=\prod_{i=1}^{n}(\sigma_i(c)z_i+\sigma_i(d))$$

The action of $SL(2,K)$ on $\mathfrak{h}^n\times\mathbb{C}^n$ is given by: Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

$$(A,(\tau,z))\longmapsto A(\tau,z):=\left(\underbrace{\frac{a\tau+b}{c\tau+d}}_{=A\tau},\ \frac{z}{c\tau+d}\right)$$

$SL(2,K)$ also acts on $\mathfrak{h}^n$ via: $(A,\tau)\mapsto A\cdot\tau:=\frac{a\tau+b}{c\tau+d}$

For fixed $k\in\mathbb{Z}$, ($m\gg 0$ later we shall use) $m\in\partial^{-1}$, we have a natural action of $SL(2,K)$ on $Hol(\mathfrak{h}^n\times\mathbb{C}^n)$ — Heisenberg group $\gamma(A,\tau)=c\tau+d$ — it's cocycle property of γ & $\det(A)=1$

$$(f,A)\longmapsto f|_k A:=f(A(\tau,z))\,e\!\left(\frac{-mcz^2}{c\tau+d}\right)\cdot N(c\tau+d)^{-k}$$

Here we have:

$e(x)=\exp(2\pi i\,tr(x))$

Cocycle property of γ function necessary:

$\gamma(AB,\tau)=\gamma(A,B\tau)\gamma(B,\tau)$

$H(\mathbb{R})$ acts on $\mathfrak{h}^n\times\mathbb{C}^n$ via:

$$(\underset{(\lambda,\mu)}{(x,r)},(\tau,z))\longmapsto(x,r)(\tau,z):=(\tau,z+\lambda\tau+\mu)$$

Then two actions define an action of $J(K) = SL(2,K) \ltimes H(K)$ on $\mathcal{G}^n \times \mathbb{C}^n$

~~(A,h)~~ ~~(B,h')~~

via:

$((A,h),(\tau,t)) \longmapsto (A,h)(\tau,t) := A(h(\tau,t))$

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VISA CHECKLIST
FOR FAMILY & FRIENDS VISIT APPLICATION

$H(R)$ acts on $\mathrm{Hol}(\mathcal{G}^n \times \mathbb{C}^n)$ via:

This checklist is provided only as a guide in obtaining a Visit visa to the UK. Supply all documentation you feel necessary to show you qualify for entry into the UK. The decision of all visa applications is at the discretion of the Entry Clearance Office. Select the correct box as it relates to the supporting documentation provided. If any of the documentation is a missing or a photocopy, denote this in the Comments area.

$(f,(X,\mu)) \longmapsto f|_{(\lambda,\mu)}(X,\mu)(\tau,z) := e^m[\lambda^2\tau + 2\lambda z + \lambda\mu] f(\tau, z+\lambda\tau+\mu)$

APPLICANT NAME: BOYLAN HATICE

List all names the Applicant is known as

$J(K)$ acts on $\mathrm{Hol}(\mathcal{G}^n \times \mathbb{C}^n)$

~~A, x~~

A, X, r

$((\cancel{A,h}), f)$

$(f,(A,h)) \longmapsto f|_{k,m}(A,h)(\tau,t) = f|_{k,m} A \,|_{k,m}\, h\,(\tau,t)$

CRUCIAL DOCUMENTS

The crucial documents are required to process a Visa Application Form

	Original	Copy	None	COMMENTS
Completed and Signed Application Form (VAF-1)	☑	☐	☐	
A valid national passport or travel document with at least one blank page, blank on both sides, and valid for at least a further 6 months	☑	☐	☐	
One current passport sized (45mm x 35mm) **colour** photograph of each applicant (front facing) taken against a white background, no sunglasses, hat, or other head covering (unless because of religious belief or ethnic background), clear and of good quality, not framed or backed, printed on normal photographic paper and not a scanned photograph.	☑	☐	☐	
A valid residence permit/Schengen multiple entry visa for Germany in your current/valid passport (resident permit with 2 months validity from expected departure date from the UK, valid Schengen visa required for any Visit Visa application from non-residents in Germany)	☑	☐	☐	
The visa fee (**Refer to the Fee Table)** or proof of online payment.	☐	☐	☐	
Children under 18: Letter of Consent from Parent/Guardian (if applicable)	☐	☐	☑	
Children under 18 - traveling with Non-Parent: Power of Attorney (if applicable)	☐	☐	☑	

RECOMMENDED DOCUMENTS

If you have these documents, you should provide them as part of your Visa Application.

	Original	Copy	None	COMMENTS
Evidence that you intend to leave the UK at the end of your visit (e.g. a letter from your employer).	☑	☐	☐	
Bank statements and payslips, or some other evidence to show that you can pay for the trip and that you have enough money to support yourself and any dependants without working or receiving help from public funds.	☐	☑	☐	BS
An up-to-date employment and leave letter.	☐	☐	☑	
If you are unemployed, provide an up-to-date letter from the Unemployment Agency confirming that you are in receipt of unemployment benefits and have permission to travel abroad.	☐	☐	☑	

Now we define the Heisenberg group of K

We denote it by $H(K) = (K \times K) . K$

The multiplication in $H(K)$ is given as:

$$(x,r)(y,s) = (x+y,\ r+s+\det\left(\binom{x}{y}\right)).$$

The Heisenberg group $H(K)$ also acts on $Hol(\mathfrak{h}^n \times \mathbb{C}^n)$ via:

$$\left(f, \underset{[\lambda,\mu]}{(x,r)}\right) \longmapsto f|_{k,m}(x,r)(\tau,z) := e(m\lambda^2\tau + 2m\lambda z + m\lambda\mu + mr) \cdot f(\tau, z+\lambda\tau+\mu)$$

These two actions in fact define an action of $SL(2,K) \ltimes H(K) =: J(K)$ on $Hol(\mathfrak{h}^n \times \mathbb{C}^n)$, where the semidirect product is given by:

$$(A,h)(A',h') = (AB, (hA').h')$$

$SL(2,K)$ acts on $H(K)$

$((x,r),A) \Rightarrow (x,r).A := (xA, r)$

$(A,(x,r))(B,(y,s)) = (AB,\ xB+y,\ r+s+\det\binom{xB}{y})$

Defn: A Jacobi of weight $k \in \mathbb{Z}$ and index $m \in \mathfrak{d}^{-1}$ (later we also assum $m \gg 0$) ($\cancel{m \in \mathfrak{d}'}$, $\cancel{m \gg 0}$) with character $\chi: SL(2,\mathcal{O}) \longrightarrow S'$ is a function where $\ker \chi \subseteq SL(2,\mathcal{O})$ finite index (χ is of finite order)

and $\phi: \mathfrak{h}^n \times \mathbb{C}^n \xrightarrow{hol} \mathbb{C}$ satisfies.

a) $\phi|_{k,m} g = \phi\, \chi(g) \quad \forall g \in J(\mathcal{O})$ | Define $J(\mathcal{O})$

~~b)~~ $\phi|_{k,m} g(\tau,z) = \sum_{t,r \in \mathfrak{d}^{-1}} c_\phi(t,r) q^t \zeta^r$, $\quad q^t(\tau) := e(t\tau)$, $\zeta^r(z) := e(rz)$

(Here $\chi(A,([\lambda,\mu],r)) = \chi(A)$)

Remark: ~~For the case $K = \mathbb{Q}$, we need to add some regularity condition at ∞.~~ $K \neq \mathbb{Q}$ we have.

Thm: (Koecher principle) $c_\phi(t,r) = 0$ unless $4mt - r^2 \geq 0$.

(proof later) (K) for $K = \mathbb{Q}$ we add this as an assum. for the deg

$J_{k,m}(\chi)$ $\mathbb{C}$ v.s. of all these functions. $\quad J_{k,m}(\chi)$, $J_{k,m}(1) =: J_{k,m}$

Ex: Later for $k = \frac{1}{2}$, we will have an example, namely:

Consider $K = \mathbb{Q}(\sqrt{17})$.

$$\vartheta^k(\tau,z) = \sum_{r \in \mathcal{O}} \left(\frac{\ }{\ }\right) q^{\frac{r^2}{8\varepsilon\sqrt{17}}} \zeta^{\frac{r}{\sqrt{17}}} \quad \text{where } \varepsilon = 4+\sqrt{17}$$

$q^{t(\tau)} := e(t\tau)$, $\zeta^{r(z)} := e(rz)$

is a ... $(\vartheta^k)^2 \in J_{1,\frac{4\varepsilon}{\sqrt{17}}}$ (a suitable χ) only one: $\varepsilon > 1$

Recall $\mathfrak{h}$ ~~equal~~ not equal the $\mathcal{O}^* = \varepsilon^{\mathbb{Z}} \times (\pm 1)$ (Dirichlet unit thm) Actually there are ex. 4 generators: $\varepsilon, \varepsilon^{-1}, -\varepsilon, -\varepsilon^{-1}$

Continue. Section 1 !

Let K be a totally real number field of degree n. $\mathcal{O}_K$ be the max. order of K.
$\mathfrak{h}^n = \underbrace{\mathfrak{h} \times \cdots \times \mathfrak{h}}_{n\text{-times}}$, where $\mathfrak{h}$ is the upper half plane]
which I defined before.

let $\sigma_i : K \hookrightarrow \mathbb{R}$ be the embeddings of K into $\mathbb{R}$.

∂_K : is the diff of K.

$$SL(2, \mathcal{O}_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_K) \;\middle|\; ad - bc = 1 \right\}$$

Let us fix an element $A \in SL_2(\mathcal{O}_K)$. $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

~~The~~ action of $SL(2, \mathcal{O})$ on $\mathfrak{h}^n$ is given by:

~~fert~~ $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O})$ and $(\tau_1, \ldots, \tau_n) \in \mathfrak{h}^n$ ~~then~~,

~~the action~~

$$A.\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \ldots, \tau_n) = \left(\cancel{\frac{a\tau_1 + b}{c\tau_1 + d}} = \frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{\sigma_n(a)\tau_n + \sigma_n(b)}{\sigma_n(c)\tau_n + \sigma_n(d)} \right)$$

$$= \cancel{\left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{\sigma_n(a)\tau_n + \sigma_n(b)}{\sigma_n(c)\tau_n + \sigma_n(d)} \right)}$$

Now the action of $SL(2, \mathcal{O}_K)$ on $\mathfrak{h}^n \times \mathbb{C}^n$ is given by: for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_K)$, $(\tau, z) \in \mathfrak{h}^n \times \mathbb{C}^n$,
where $\tau = (\tau_1, \ldots, \tau_n)$
$z = (z_1, \ldots, z_n)$.

$$A \cdot (\tau, z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau, z) = \cancel{(A\tau, Az)}$$

$$= \cancel{(A\tau} \quad A\tau,$$

$$= \left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{\sigma_n(a)\tau_n + \sigma_n(b)}{\sigma_n(c)\tau_n + \sigma_n(d)} ; \frac{z_1}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{z_n}{\sigma_n(c)\tau_n + \sigma_n(d)} \right)$$

For fixed $k \in \mathbb{Z}$, $m \in \mathcal{O}^{-1}$, we have a "natural"
~~The~~ action of $SL(2, \mathcal{O}_K)$ on $Hol(\mathfrak{h}^n \times \mathbb{C}^n)$ ~~is given by~~
(Let $m \in \partial_K^{-1}$, $(m \gg 0$, $k \in \mathbb{Z}$)

$(f, A) \mapsto f|_{k,m} A \in Hol(\mathfrak{h}^n \times \mathbb{C}^n)$, where
for $A \in SL_2(\mathcal{O}_K)$
$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(f|_{k,m} A)(\tau, z) = f(A \cdot (\tau, z)) \cdot e\left(-\sum_{i=1}^{n} \frac{\sigma_i(mc) z_i^2}{\sigma_i(c)\tau_i + \sigma_i(d)}\right) \cdot \prod_{j=1}^{n} (\sigma_j(c)\tau_j + \sigma_j(d))^{-k} \cdot \cancel{w_A(z_1 \ldots z_n)}^{-1}$$

$(\tau = (\tau_1, \ldots, \tau_n)$
$z = (z_1, \ldots, z_n))$

~~where~~ ~~$w_A : \mathfrak{h}^n \to \mathbb{C}$~~

[Here $e(*) = \exp(2\pi i *)$]

For fixed $(k \in \mathbb{Z})$ and $m \in \mathfrak{d}^{-1}$, the action of $\Theta_K \times \Theta_K$ on $\mathrm{Hol}(\mathfrak{h}^n \times \mathbb{C}^n)$

is given by:

for $f \in \mathrm{Hol}(\mathfrak{h}^n \times \mathbb{C}^n)$, $(f, [\lambda, \mu]) \mapsto f|_{k,m}[\lambda, \mu]$, where

$$f|_{k,m}[\lambda, \mu](\vec{\tau}, \vec{z}) = f(\vec{\tau}; \vec{z} + \lambda\vec{\tau} + \mu)\, e\left(\sum_{i=1}^{n} \sigma_i(\lambda^2 m)\tau_i + \sigma_i(2m\lambda)z_i\right)$$

$\lambda, \mu \in \Theta_K$

where $\vec{z} + \lambda\vec{\tau} + \mu := (z_1 + \sigma_1(\lambda)\tau_1 + \sigma_1(\mu), \dots, z_n + \sigma_n(\lambda)\tau_n + \sigma_n(\mu))$

$\vec{\tau} = (\tau_1, \dots, \tau_n)$

$\vec{z} = (z_1, \dots, z_n)$

These two actions define in fact an action of $SL(2, \mathcal{O}) \ltimes \mathcal{O} \times \mathcal{O} =: SL(2,\mathcal{O})^J$ on $\mathrm{Hol}(\mathfrak{h}^n \times \mathbb{C}^n)$, where this semidirect product is defined by:

$(A, [\lambda, \mu])(A', [\lambda', \mu']) = AA', [\lambda, \mu]A' \dots$

Defn: ~~Now we can define~~ ~~the~~ Jacobi form of weight $k \in$ ~~$\mathbb{Z}$~~ and index $m \in \mathfrak{d}^{-1}$ ~~and~~ with character χ ($\chi : SL_2(\mathcal{O}_K) \to S^1$) over ~~totally real number field~~ K

is a function

$$\phi : \mathfrak{h}^n \times \mathbb{C}^n \xrightarrow{\text{hol.}} \mathbb{C} \quad \text{satisfying}$$

(1) $\phi|_{k,m}(\vec{\tau}, \vec{z}) = \phi(\vec{\tau}, \vec{z}) \cdot \chi(g)$ for all $g \in SL(2, \mathcal{O})^J$

Remark: For $n = 1$ we need an additional regularity condition at ∞.

~~$\phi|_{k,m}[\lambda, \mu](\vec{\tau}, \vec{z}) = \phi(\vec{\tau}, \vec{z})$~~.

~~$\lambda, \mu \in \Theta_K$~~

(here $\chi((A, [\lambda, \mu])) := \chi(A)$)

$J^{k}_{k,m}(\chi)$:= $\mathbb{C}$-vector space of all these functions.

Remark: Later for $k = \frac{1}{2}$ we'll see we have a Jacobi form over $K = \mathbb{Q}(\sqrt{17})$ given by

~~we have a Jacobi form~~

~~$\vartheta^K = \sum_{r \in \mathcal{O}_K} \left(\frac{-4}{N(r)}\right) \exp\left(2\pi i \left(\sum_{i=1}^{n} \frac{\sigma_i(r^2)\tau_i}{8\sigma_i(\omega)} + \frac{\sigma_i(r)}{\sigma_i(\omega)} z_i\right)\right)$~~

$$\vartheta^K = \sum_{r \in \mathcal{O}_K} \chi(r) \left(\frac{-4}{Nr}\right) \cdot \exp\left(\sum_{i=1}^{2} \frac{\sigma_i(r^2)}{8\sigma_i(\varepsilon\sqrt{17})}\tau + \frac{\sigma_i(r)\, z_i}{\sigma_i(\sqrt{17})}\right)$$

where ε is the fundamental unit > 1.

$\varepsilon = \frac{4 + \sqrt{17}}{2}$, $N(\varepsilon) = -1$, $\varepsilon > 1$