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Singular Jacobi Forms over # fields : (1)

Main interest comes from the fact that we expect many theorems from the theory of elliptic modular forms and elliptic Jacobi forms hold true for the number field situation, too.

In particular, there should be a lifting from Jacobi forms to Hilbert modular forms and the Fourier coefficients of Jacobi forms over number fields should encode the vanishing of L-function associated to Hilbert modular forms at the critical point. This is the ultimate goal to prove such theorems. For the moment it is already interesting to produce explicit examples of Jacobi forms over number fields. That is the main goal of my talk. So, to begin with I start to introduce a lot of notations unfortunately.

Let K be a totally real # field of degree n.

∂: different of K $\qquad e(x) = e^{2\pi i\, tr(x)}$

$\mathcal{O}$: ring of integers of K

$\sigma_i : K \hookrightarrow \mathbb{R} \quad (i=1..n)$

$\mathbb{C}^n \simeq C := \mathbb{C} \underset{\mathbb{Q}}{\otimes} K \quad , \quad R = \mathbb{R} \underset{\mathbb{Q}}{\otimes} K$

$(\sigma_i(a)) \mapsto z \otimes a$

C becomes a ring and an algebra over $\mathbb{C}$ and K.

$\mathcal{H} = \{ \tau \in C \mid Im(\sigma_i(\tau)) > 0 \quad \forall i = 1..n \}$

Here $\sigma_i \ (i=1..n)$ is extended $\mathbb{C}$-linearly to C.

$N(c) = \prod_{i=1}^{n} \sigma_i(c) \quad , \quad tr(\tau) = \sum_{i=1}^{n} \sigma_i(c) \qquad (c \in C)$.

$SL(2,K) \subseteq SL(2,R)$ act on $\mathcal{H}$

<u>Defn:</u> ($\mathcal{O}$-lattice) An ($\mathcal{O}$-lattice) integral lattice over $\mathcal{O}$ is a pair $\underline{L} = (L, \beta)$, where L denotes a f.g. torsion-free $\mathcal{O}$-module and $\beta : L \times L \to \partial^{-1}$

is a map which satisfies

i) β is non-degenerate ($\beta(x,L)=\{0\} \iff x=0$)

ii) The map β is $\mathcal{O}$-bilinear and symmetric.

An $\mathcal{O}$-lattice is called even if $\beta(x,x) \in 2\partial^{-1}$ $(x \in L)$.

" " " " totally-positive-definite if

$\sigma_i(\beta(x,x)) > 0$ $(x \in L, x \neq 0, i=1..n)$.

The rank of an $\mathcal{O}$-lattice L is the dimension of the $K \otimes_{\mathcal{O}} L$ as a K-vector space.

The dual of L is denoted by $L^{\#}$ is defined as

$$L^{\#} = \{ x \in K \otimes_{\mathcal{O}} L \mid \beta(x,L) \in \partial^{-1} \}$$

Here β is extended K-linearly to $K \otimes_{\mathcal{O}} L$.

We denote $L_{\mathbb{C}} := \mathbb{C} \otimes_{\mathcal{O}} L$. The space $L_{\mathbb{C}}$ is a $\mathbb{C}$-algebra containing L and K as $\mathcal{O}$-submodules.

Defn: (Metaplectic cover of $SL(2,\mathcal{O})$)

$$Mp(2,\mathcal{O}) = \{ (A,\omega) \mid A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathcal{O}),\ \omega : \mathbb{H} \xrightarrow{hol} \mathbb{C},\ \omega^2(\tau) = N(c\tau+d) \}$$

$Mp(2,\mathcal{O})$ becomes a group via:

$$(A,\omega)(B,\nu) = (AB, \omega(B\tau)\nu(\tau)).$$

Defn: (Heisenberg group)

The Heisenberg group associated to an $\mathcal{O}$-lattice L is defined as:

$$H(L_{\mathbb{R}}) = \{ (x,y,\xi) \mid x,y \in L_{\mathbb{R}},\ \xi \in S^1 \}$$

$L_{\mathbb{R}} = \mathbb{R} \otimes_{\mathcal{O}} L$

with the operation:

$$(x,y,\xi)(x',y',\xi') = (x+x', y+y', \xi\xi' e(\tfrac{1}{2}\beta(x,y') - \tfrac{1}{2}\beta(x',y)))$$

$H(L)$ is a subgroup of $H(L_{\mathbb{R}})$ generated by the elements $(x,0,1)$, $(0,y,1)$ $(x,y \in L)$.

<u>Defn:</u> (Jacobi forms) Let $\underline{L} = (L,\beta)$ be a totally positive definite even $\mathcal{O}$-lattice and $k \in \frac{1}{2}\mathbb{Z}$. Moreover, $\chi : SL(2,\mathcal{O}) \to S^1$ denotes a character on $SL(2,\mathcal{O})$ with $[SL(2,\mathcal{O}) : \ker \chi] < \infty$.

A Jacobi form over K of weight k and index $\underline{L}$ and character χ on $Mp(2,\mathcal{O})$ is a holomorphic fn. $\phi : H \times L_{\mathbb{C}} \longrightarrow \mathbb{C}$ satisfying

i) $\phi|_{k,\underline{L}} (A,\omega)(\tau,z) := \phi\left(A\tau, \frac{z}{c\tau+d}\right) e\left(\frac{-c\beta(z)}{c\tau+d}\right) \omega(\tau)^{-2k} = \chi(A)\, \phi(\tau,z)$

ii) $\phi|_{k,\underline{L}} \underset{(x,y,\zeta)}{h} (\tau,z) := \zeta\, \phi(\tau, z+x\tau+y)\, e(\tau\beta(x)+\beta(x,z)+\frac{1}{2}\beta(x,y)) = \phi(\tau,z)$ $\quad ((A,\omega) \in Mp(2,\mathcal{O}))$, $x,y \in L$, $\zeta \in S^1$

$h \in H(L)$ = Heisenberg group associated to an $\mathcal{O}$-lattice $\underline{L}$.

The space of such functions is denoted by $J_{k,\underline{L}}(\chi)$.

Note that $\dim J_{k,\underline{L}}(\chi) < \infty$

$= 0$ for $k \leq \frac{r}{2}$ $\quad (r = \operatorname{rank}(\underline{L}))$

$k = \frac{r}{2}$: singular +

$k = \frac{r}{2}+1$: critical

In my talk I classify all singular JF of index of a rank 2 lattice.

<u>Main Theorem</u> : (B) Suppose 2 splits completely in K and f be the (possibly) empty product of distinct prime ideals above 2 of degree 1. Let t be an integral s.t. $t^2\omega\partial = 4f$ $(\omega \in K^*$, $\omega\partial > 0)$. We set

$$\vartheta_{(\mathfrak{c},\omega)}(\tau,z) = \sum_{r\in(\omega\partial\mathfrak{c})^{-1}} \left(\frac{-4}{x}\right)\left(\frac{x}{\mathfrak{f}}\right) q^{\frac{1}{2}\omega r^2}\, e(\omega r z)\,. \qquad \binom{\tau\in H}{z\in L_{\mathfrak{c}}}$$ (4)

Here $x \equiv r x_0^{-1} \bmod \mathfrak{c}$ and $(\omega\partial\mathfrak{c})^{-1} = \mathcal{O}x_0 + \mathfrak{c}$.

Then, $J_{\frac{1}{2},(\mathfrak{c},\omega)}(\chi_\omega) = \mathbb{C}\,\vartheta_{(\mathfrak{c},\omega)}$ where

$\chi_\omega\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = e\left(\frac{b\omega x^2}{2}\right)$ for some x s.t. $\left(\frac{-4}{x}\right)\left(\frac{x}{\mathfrak{f}}\right) \neq 0$ ~~for some x relatively prime~~

~~One~~ can show ~~that~~ $S(\mathcal{O}_\omega)^{ab}$ is generated by

$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ $(b\in\mathcal{O})$. Hence χ_ω is uniquely determined by

$\chi_\omega\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$ $(b\in\mathcal{O})$.

Here $(\mathfrak{c},\omega)$ denotes the lattice $(\mathfrak{c},(x,y)\mapsto\omega xy)$.

Moreover $\left(\frac{-4}{x}\right)$: ~~nonzero~~ DC modulo 4

$\left(\frac{x}{\mathfrak{f}}\right)$: product of nonzero DC modulo prime

ideals above 3 of degree 1.

Main Theorem 2 (B)

Let $\underline{L} = (L,\beta)$ be a totally positive definite even $\mathcal{O}$-lattice. Then $J_{\frac{1}{2},\underline{L}}(\chi) = 0$, unless $\underline{L}$ can be embedded into some lattice $(\mathfrak{c},\omega)$ and $\chi = \chi_\omega$ as in the previous theorem.

Then $J_{\frac{1}{2},\underline{L}}(\chi) = \alpha^* J_{\frac{1}{2},(\mathfrak{c},\omega)}(\chi_\omega)$. Here α^* denotes the pullback of the embedding. ($\alpha : \underline{L} \to (\mathfrak{c},\omega)$ then $\alpha^*\phi(\tau,z) = \phi(\tau,\alpha(z))$)

How to obtain the main result:

In the following $\underline{L} = (L,\beta)$ denotes always a totally positive definite even $\mathcal{O}$-lattice of rank r.

Basic Theta functions:

$$\vartheta_{\underline{L},x}(\tau,z) = \sum_{\substack{r\in L^\# \\ r\equiv x \\ (L)}} q^{p(r)}\, e(\beta(r,z)) \quad (x\in L^\#/L)$$

$$(\tau\in H,\ z\in L_{\mathbb{C}})$$

Here $q^t(\tau) := e(t\tau)$.

$\vartheta_{\underline{L},x}$ are singular Jacobi forms of weight $\frac{r}{2}$ on some subgroup of $Mp(2,\mathbb{Z})$.

Theorem: (Theta expansion) Let $\phi \in J_{k,\underline{L}}(\chi)$.

Then,
$$\phi(\tau,z) = \sum_{x\in L^\#/L} h_x(\tau)\, \vartheta_{\underline{L},x}(\tau,z)$$

h_x are hol. fns. on H (in fact, HMF).

Using this result we see that there are no JF of weight $k<\frac{r}{2}$ since there are no JF of negative weight. So, the first interesting case is $k=\frac{r}{2}$ (singular) that we are interested in.

So, from theta-expansion we see that singular JF is a linear combination of basic theta functions. So, to classify all singular forms we need to study the space spanned by the basic theta functions. We denote

$$\Theta_{\underline{L}} = \mathrm{span}_{\mathbb{C}} \langle \vartheta_{\underline{L},x} \; ; \; x\in L^\#/L \rangle .$$

Θ_L becomes a $G:=Mp(2,\mathbb{Z})$-module via $|_{\frac{r}{2},\underline{L}}$-action. It is easy to see that singular JF are in one-to-correspondence with the one-dim. G-submodules of $\Theta_{\underline{L}}$.

So, we study the one-dimensional G-submodules of $\Theta_{\underline{L}}$. For that we use the theory of weil Representations since there is a one-to-one correspondence between them, namely,

$$W(D_{\underline{L}}^{-1}) \xrightarrow{\sim} \Theta_{\underline{L}}^{\vee} \text{ as left } G\text{-modules.}$$

$\hookrightarrow \Theta_{\underline{L}}$ with the G-action:

$$(g, \vartheta_{\underline{L},x}) \mapsto g\vartheta_{\underline{L},x} := \vartheta|g^{-1}_{\frac{k}{2},\underline{L}}$$

<u>Defn:</u> (fq $\mathcal{O}$-module)

A fq $\mathcal{O}$-module is a pair (M, Q), where M is a finite ~~abelian group~~ $\mathcal{O}$-module and Q is a qf on M, i.e. $Q: M \to K/\mathfrak{d}^{-1}$ is such that

i) $Q(ax) = a^2 Q(x)$ $\quad (a \in \mathcal{O},\ x \in M)$

ii) The map $B: M \times M \to \mathfrak{d}^{-1}$ is defined by $B(x,y) := Q(x+y) - Q(x) - Q(y)$ is $\mathcal{O}$-bilinear and symmetric.

iii) B is non-degenerate $(B(x, M) = \{0\} \Leftrightarrow x = 0)$

<u>Exp:</u> $D_L = \left(L^{\#}/L,\ x+L \mapsto \frac{1}{2}\beta(x,x) + \mathfrak{d}^{-1}\right)$ ~~for~~ $(\underline{L} = (L, \beta))$ defines an fq $\mathcal{O}$-module. It is known as Discriminant module associated to $\underline{L}$.

$W(M)$: The G-module $\mathbb{C}[M]$ ($\mathbb{C}$ v.s. of functions from M into $\mathbb{C}$ the basis elements are e_x $(x \in M)$) where left G-action is given by

$M = L^{\#}/L$, $Q(x) = \frac{1}{2}\beta(x,x)$, $\sigma(M) = \sigma(L)$

$$(g, e_x) \mapsto g e_x := \rho(g)(e_x)$$

$\rho: G \to GL(\mathbb{C}[M])$ is the representation (weil) defined by

$$\rho\left(\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right) e_x = e(b\,Q(x))\, e_x$$

$$\rho\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \sqrt{\tau}\right) e_x = \sigma(M) \frac{1}{\sqrt{|M|}} \sum_{y \in M} e(-B(x,y))\, e_y$$

$$\rho\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, -1\right) e_x = (-1)^{?} e_x$$

$$\sigma(\mathcal{D}_L) = \frac{1}{\sqrt{|M|}} \sum_{y \in M} e(-Q(x)) .$$

So, if we can decompose $W(\mathcal{D}_L)$ then we can also decompose Θ_L using the isomorphism above.
As I mentioned before I consider rank 1 lattices. But they have cyclic discriminant module. (M is cyclic if $M = \mathbb{Z}x_0$ for some $x_0 \in M$)
and we can decompose $W(\mathcal{D}_L)$ into (M, Q) irreducible G-submodules: in this case:

$$W(\mathcal{D}_L) = \bigoplus_{\substack{b^2 \mid \frac{a^2}{\ell}}} \ \bigoplus_{\substack{f \mid \frac{a^2}{b^2 \ell} \\ f \text{ square-free}}} i_{U_f}(W(\mathcal{D}_{L}/U_f)^{\text{new}, f})$$

← dimension formula for these new irr. space.

↑ certain irreducible G-submodules of $W(\mathcal{D}_L)$

a: exp of M
ℓ: level of M

(– taking quotients
– f: parametrizes the characters of the orthogonal group of $\mathcal{D}_L$.
the action of the orthogonal group on $\mathbb{C}[\mathcal{D}_L]$ commutes with G-action
so, the eigenspaces of f becomes a G-module
we decompose into these eigenspaces)

$\pi: L^\# \to L^\#/L$
$U \subseteq L^\#/L$ some certain θ-nul
$L/U = (\pi^{-1}(U), \beta)$

Thm:

$$\begin{array}{ccc} W(\mathcal{D}_{L/U}^{-1}) & \xrightarrow{\sim} & \Theta_{L/U} \\ \downarrow i_U & & \downarrow \\ W(\mathcal{D}_L^{-1}) & \xrightarrow{\sim} & \Theta_L \end{array}$$

$W(\mathcal{D}_{L/U})^{-1} \cong$

Hence, we can decompose Θ_L when L has rank 1

$$\Theta_L = \bigoplus_{\substack{b^2 \mid \frac{a^2}{\ell}}} \ \bigoplus_{\substack{f \mid \frac{a^2}{b^2 \ell} \\ f \text{ square-free}}} \Theta_{L/U}^{\text{new}, f}$$

a: exp of $\mathcal{D}_L$
ℓ: level of $\mathcal{D}_L$.