Appendix to Cris Poor and David Yuen's article by Tomoyoshi Ibukiyama and Nils-Peter Skoruppa

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Non-existence of Siegel modular forms of weight one

For any natural number N, we denote by $\Gamma_0(N)$ the congruence subgroup of $Sp(2,\mathbb{Z})$ defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in Sp(2,\mathbb{Z}); A, B, C, D \in M_2(\mathbb{Z}) \right\}.$$

The purpose of this appendix is to prove

Theorem 1. *For any natural number N, we have $S_1(\Gamma_0(N)) = 0$.*

The proof of this theorem is given by using results on Jacobi forms of degree one, so we explain this first.

Let $J_{1,m}(l)$ denote the space of Jacobi forms of index m on $\Gamma_0(l) \ltimes \mathbb{Z}^2$. We shall prove the following theorem in the next section.

Theorem 2. *Let m, l_1, l_2 be positive integers such that $l_1|m^\infty$ and l_2 is relatively prime to m. Then $J_{1,m}(l_1l_2) = J_{1,m}(l_1)$.*

In [Sko, Satz 6.1] it was proven that $J_{1,m}(1) = 0$ for all m. Thus, the above theorem has an immediate consequence.

Corollary 1. $J_{1,m}(l) = 0$ *for all l and m which are relatively prime.*

Now, admitting this theorem for a while, we give a proof of Theorem 1. The proof relies on two easy observations. Take $F \in S_1(\Gamma_0(N))$ and write the Fourier expansion as

$$F(Z) = \sum_{T \in L_+^*} a(T)e(\mathrm{tr}(TZ))$$

where T runs over the set L_+^* of all half-integral positive definite symmetric matrices and $e(x) = e^{2\pi ix}$. For $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} \in L_+^*$, we denote by $\mathrm{cont}(T)$ the greatest common divisor of m, n, r. We first show

Lemma 1. *Notations being as above, we have* $a(T) = 0$ *if* $(\mathrm{cont}(T), N) = 1$.

Proof. For a variable Z in the Siegel upper half space H_2 of degree two, we write $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix}$. We put

$$f_m(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}, \\ 4nm-r^2>0}} a\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} e(n\tau + rz).$$

Then $F(Z) = \sum_{m=1}^{\infty} f_m(\tau, z)\, e(m\omega)$ gives the Fourier-Jacobi expansion of F and $f_m(\tau, z) \in J_{1,m}(\Gamma_0(N))$. By Corollary 1, we have $f_m = 0$ if m is prime to N. Hence we have $a(T) = 0$ if the (2,2)-component of T is prime to N. Now there exists an integral vector $v \in \mathbb{Z}^2$ such that ${}^t vTv/\mathrm{cont}(T)$ is prime to N. (This is classically well known and easy to see. For example see [Zagier].) This implies that there exists $U \in GL_2(\mathbb{Z})$ such that the $(2, 2)$ component of ${}^tUTU/\mathrm{cont}(T)$ is prime to N. By the automorphic property of F, we have $F(UZ{}^tU) = \det(U)F(Z)$ for any $U \in GL_2(\mathbb{Z})$. This implies that $a({}^tUTU) = \det(U)a(T)$ for any $U \in GL_2(\mathbb{Z})$. Hence we have $a(T) = 0$ if $\mathrm{cont}(T)$ is coprime to N and we have proven the lemma. □

Secondly, we use standard Hecke operators at bad primes. Let m be any natural number such that if $p|m$ then $p|N$. We put

$$U(m) = \Gamma_0(N) \begin{pmatrix} 1_2 & 0 \\ 0 & m1_2 \end{pmatrix} \Gamma_0(N),$$

where 1_2 is the unit matrix of size two. Then we have

$$U(m) = \coprod_S \Gamma_0(N) \begin{pmatrix} 1_2 & S \\ 0 & m1_2 \end{pmatrix}$$

where S runs over a complete set of representatives of integral symmetric matrices module m. This is easily shown as follows. We may assume that $N \neq 1$. We put

$$\Gamma_0(N, m) = \Gamma_0(N) \cap \begin{pmatrix} 1_2 & 0 \\ 0 & m1_2 \end{pmatrix}^{-1} \Gamma_0(N) \begin{pmatrix} 1_2 & 0 \\ 0 & m1_2 \end{pmatrix},$$

i.e.

$$\Gamma_0(N, m) = \left\{ \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in \Gamma_0(N) : B \equiv 0 \bmod m. \right\}$$

For any $\gamma = \begin{pmatrix} A & B \\ NC & D \end{pmatrix} \in \Gamma_0(N)$, we have $A^tD - NB^tC = 1_2$ so $\det(A)$ is prime to N and also to m. Since $A^{-1}B$ is symmetric, there exists an integral symmetric matrix S such that $AS + B \equiv 0 \bmod m$. Hence we have

$$\gamma \in \Gamma_0(N,m) \begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix}.$$

Hence the coset decomposition of $U(m)$ is given as above.

Now we can define an action of $U(m)$ on $S_1(\Gamma_0(N))$ by

$$F|U(m) = m^{-3} \sum_{S={}^tS \in M_2(\mathbb{Z}/m\mathbb{Z})} F((Z+S)/m) = \sum_{T \in L_+^*} a(mT) e(\mathrm{tr}(TZ)).$$

Since this is the usual action in the theory of Hecke operators, we see $F|U(m) \in S_1(\Gamma_0(N))$. Now for a fixed T, take a natural number m such that $m|N^\infty$ and $\mathrm{cont}(T/m)$ is coprime to N. Put $T_0 = T/m$. Then $a(T) = a(mT_0)$ is the coefficient of $F|U(m)$ at T_0. But $\mathrm{cont}(T_0)$ is coprime to N. Hence we have $a(T) = 0$ by Lemma 1. So all the coefficients of F vanish and hence $F = 0$. Thus we have proven Theorem 1.

Jacobi forms of weight one on $\Gamma_0(l)$

The proof of Theorem 2 relies on the explicit description of the irreducible components of the (projective) $\mathrm{SL}(2,\mathbb{Z})$-module of modular forms of weight $\frac{1}{2}$ on all principal congruence subgroups $\Gamma(4m)$. This was given in [Sko, Satz 5.2] based on the theorem of Serre-Stark [Ser-Sta] that there are no modular forms of weight $\frac{1}{2}$ on any $\Gamma_0(l)$ except for theta series.

The proof of the Theorem 2 yields actually a formula for the dimension of the spaces $J_{k,m}(l)$ for arbitrary l and m, which, for given instances of l and m, can be explicitly evaluated. However, putting this formula in a more explicit form would require a deeper understanding of the rings of virtual characters of the groups $\mathrm{SL}(2,\mathbb{Z}_p)$. We hope to come back to this in another place.

Though it is very likely that the assumption on l, m is not best possible, it is in any case not superfluous. In fact, there exists Jacobi forms of weight 1. For instance, it is not difficult to show that

$$\sum_{x,y \in \mathbb{Z}} \left(\frac{2x+y}{7}\right) q^{x^2+xy+2y^2} \zeta^{7y} \in J_{1,7}(49).$$

A more detailed treatment of the representation theory of $\mathrm{SL}(2,\mathbb{Z})$ used in the proof of Theorem 2 will be given in a broader context in [Sko2].

Proof of Theorem 2. Let $\theta_{m,\rho} = \sum_{r\equiv\rho \bmod 2m} q^{r^2/4m}\zeta^r$, let Th_m the vector space spanned by the $\theta_{m,\rho}$ $(0 \le \rho < 2m)$, and let $M_{1/2} = \sum_{m\ge 1} M_{1/2}(\Gamma(4m))$ the space spanned by all modular forms of weight $\frac{1}{2}$ which are invariant under some principal congruence subgroup $\Gamma(4m)$. The spaces Th_m and $M_{1/2}$ are $\mathrm{M}(2,\mathbb{Z})$-right modules. Here $\mathrm{M}(2,\mathbb{Z})$ denotes the usual metaplectic double cover of $\mathrm{SL}(2,\mathbb{Z})$, i.e. the group of all pairs (A,v), where $A = [a,b;c,d]$ is in $\mathrm{SL}(2,\mathbb{Z})$ and v is a holomorphic function on the complex upper half plane satisfying $v^2(\tau) = c\tau + d$, the product of two such pairs being defined by $(A,v)(B,w) = (AB, v(B\tau)w(\tau))$. The action on elements $\theta \in Th_m$ is given by $\theta|(A,v)(\tau,z) = \theta(A\tau,\, z/(c\tau+d))\exp(-2\pi imcz^2/(c\tau+d))/v(\tau)$, and the action on functions $h \in M_{1/2}$ is defined by $h|(A,v)(\tau) = h(A\tau)/v(\tau)$. Note that the transformation law for the functions $h \in M_{1/2}$ reads $h|\alpha = h$ for all $\alpha \in \Gamma(4m)^*$, where $\Gamma(4m)^*$ is the subgroup of all pairs $(A, j(A,\tau))$ in $\mathrm{M}(2,\mathbb{Z})$ with $j(A,\tau) = \theta(A\tau)/\theta(\tau)$ and with $\theta = \sum_{r\in\mathbb{Z}} q^{r^2}$. It is not hard to show that $\Gamma(4m)^*$ is a normal subgroup of $\mathrm{M}(2,\mathbb{Z})$. The group $\Gamma(4m)^*$ acts trivially on Th_m, i.e. the action of $\mathrm{M}(2,\mathbb{Z})$ on Th_m factors through an action of the finite group $\mathrm{M}(2,\mathbb{Z})/\Gamma(4m)^*$ [Sko, Lemma 1.2]

For every subgroup Γ of finite index in $\mathrm{SL}(2,\mathbb{Z})$ the map $h\otimes\theta \mapsto h\theta$ induces an isomorphism

$$(M_{\frac{1}{2}} \otimes Th_m)^\Gamma \to J_{1,m}(\Gamma).$$

Here the space on the left is the subspace of all elements in the $\mathrm{M}(2,\mathbb{Z})$-module $M_{1/2}\otimes Th_m$ which are invariant under Γ. Note that the action on this tensor product factors through an action on $\mathrm{SL}(2,\mathbb{Z})$. We shall use this tacitly in the following. In particular, it makes sense to consider Γ-invariant elements.

For a more detailed exposition of the facts described in the two preceding paragraphs we refer the reader to [Sko-Zag, §0].

For a nonnegative integer d, the application $U_d : \theta(\tau,z) \mapsto \theta(\tau,dz)$ defines a $\mathrm{M}(2,\mathbb{Z})$-equivariant map $Th_m \to Th_{md^2}$ if $d \ge 1$ and $Th_m \to M_{1/2}(\Gamma(4m))$ if $d = 0$. There is exactly one $\mathrm{M}(2,\mathbb{Z})$-invariant complement C of the sum of the spaces Th_{m/d^2} $(d^2|m,\, d>1)$ in Th_m. For a square free divisor f of m, denote by Th_m^f the subspace of all $\theta \in C$ such that $\theta|\alpha = \left(\frac{4m}{a}\right)\mu(\gcd(f, \frac{a+1}{2}))\theta$ for all $\alpha = (A, j(A,\tau))$ in $\mathrm{M}(2,\mathbb{Z})$ with $A \equiv [a,0;0,a] \bmod 4m$, where μ is the Moebius function. The space Th_m^f is an irreducible $\mathrm{M}(2,\mathbb{Z})$-module [Sko, Satz 1.8]. The decomposition of Th_m and $M_{1/2}$ into irreducible $\mathrm{M}(2,\mathbb{Z})$-modules can now be described in terms of the Th_m^f as follows [Sko, Satz 1.8, Satz 5.2]:

$$Th_m = \bigoplus_{fd^2|m} Th^f_{m/d^2}|U_d, \quad M_{\frac{1}{2}} = \bigoplus_{g|n,\,\mu(g)=1} Th^g_n|U_0.$$

Here the first sum is over all pairs of positive integers f, d with square free f such that $fd^2|m$, the second one runs over all positive integers g, n with $\mu(g) = 1$ such that $g|n$. It is easily checked that $Th^g_n|U_0$ is isomorphic to Th^g_n as $\mathrm{M}(2,\mathbb{Z})$-module if $\mu(g) = 1$ and reduces to 0 otherwise.

Let θ^f_m denote the character of the $\mathrm{M}(2,\mathbb{Z})$-module Th^f_m. We deduce from the preceding discussion that

$$\dim J_{1,m}(\Gamma) = \sum_{\substack{f,d,g,n \\ fd^2|m,\, g|n,\, \mu(g)=1}} \langle 1_\Gamma, \mathrm{Res}_\Gamma\, \theta^f_{m/d^2}\, \theta^g_n\rangle.$$

Here, for characters χ, ψ of representations of a group G we use $\langle\chi,\psi\rangle$ for the $\mathbb{Z}$-bilinear form on the ring of virtual characters which, for irreducible χ, ψ, equals 1 or 0 accordingly as $\chi = \psi$ or not. This make sense if G is a finite group as well as for characters of representations of subgroups of $\mathrm{SL}(2,\mathbb{Z})$ whose kernel contains some normal subgroup of finite index in $\mathrm{SL}(2,\mathbb{Z})$. We remark that it is not difficult to show that in the last formula for the dimension of $J_{k,m}(\Gamma)$ the terms with $\mu(f) = +1$ vanish if $-1 \in \Gamma$. However, we shall not make use of this. Clearly, since the dimension of $J_{1,m}(\Gamma)$ is finite there are only finitely many n in the dimension formula with multiplicity $\langle 1_\Gamma, \mathrm{Res}_\Gamma\, \theta^f_{m/d^2}\, \theta^g_n\rangle$ different from 0.

To study these multiplicities we decompose them into local components. More precisely, we note that, for each m, the natural map

$$\mathrm{M}(2,\mathbb{Z})/\Gamma(4m)^* \to \mathrm{M}(2,\mathbb{Z})/\Gamma(4m_2)^* \times \prod_{p|m,\, p\neq 2} \mathrm{SL}(2,\mathbb{Z})/\Gamma(m_p)$$

is an isomorphism. Here, for a prime p, we use m_p for the exact power of p dividing m. Accordingly every irreducible character χ of a representation of $\mathrm{M}(2,\mathbb{Z})$ which is trivial on some $\Gamma(4m)^*$ admits a unique factorization $\chi = \prod_p \chi_p$ with irreducible characters χ_p each of which is the character of a representation of $\mathrm{M}(2,\mathbb{Z})$ respectively $\mathrm{SL}(2,\mathbb{Z})$ which is trivial on $\Gamma(4m_2)^*$

respectively $\Gamma(m_p)$ accordingly as $p = 2$ or $p > 2$. More generally, a corresponding factorization holds true for characters of representations of subgroups of $\mathrm{M}(2,\mathbb{Z})$ which are trivial on some $\Gamma(4m)^*$.

If Γ is a congruence subgroup we can therefore write

$$\langle 1_\Gamma, \mathrm{Res}_\Gamma\, \theta_m^f\, \theta_n^g\rangle = \prod_p \langle 1_{\Gamma_p}, \mathrm{Res}_{\Gamma_p} (\theta_m^f)_p\, (\theta_n^g)_p\rangle.$$

Here Γ_p is the inverse image of the reduction of Γ module sufficiently big powers of p. Using Frobenius reciprocity we can constrain ourself to characters of $\mathrm{SL}(2,\mathbb{Z})$. Namely, for each p, we have

$$\langle 1_{\Gamma_p}, \mathrm{Res}_{\Gamma_p} (\theta_m^f)_p\, (\theta_n^g)_p\rangle = \langle \mathrm{Ind}_{\Gamma_p}^{\mathrm{SL}(2,\mathbb{Z})} 1|_{\Gamma_p}, (\theta_m^f)_p\, (\theta_n^g)_p\rangle.$$

We now consider the case $\Gamma = \Gamma_0(l)$. Then $\Gamma_p = \Gamma_0(p^\lambda)$ where $p^\lambda \| l$. The decomposition of the trivial character of $\Gamma_0(p^\lambda)$ induced to $\mathrm{SL}(2,\mathbb{Z})$ into irreducible characters is given by

$$\mathrm{Ind}_{\Gamma_0(p^\lambda)}^{\mathrm{SL}(2,\mathbb{Z})} 1 = \psi_{p^\lambda} + \psi_{p^{\lambda-1}} + \cdots + 1, \quad \psi_{p^\lambda} = \mathrm{Ind}_{\Gamma_0(p^\lambda)}^{\mathrm{SL}(2,\mathbb{Z})} 1 - \mathrm{Ind}_{\Gamma_0(p^{\lambda-1})}^{\mathrm{SL}(2,\mathbb{Z})} 1.$$

Indeed, if we realize $\mathrm{Ind}_{\Gamma_0(p^k)}^{\mathrm{SL}(2,\mathbb{Z})} 1$ by $\mathbb{C}[\Gamma_0(p^k)\backslash \mathrm{SL}(2,\mathbb{Z})]$, considered as $\mathrm{SL}(2,\mathbb{Z})$ module via multiplication from the right, then ψ_{p^k} is the character of the kernel of the natural map

$$\mathbb{C}[\Gamma_0(p^k)\backslash \mathrm{SL}(2,\mathbb{Z})] \to \mathbb{C}[\Gamma_0(p^{k-1})\backslash \mathrm{SL}(2,\mathbb{Z})].$$

In particular, it is a proper character. Moreover,the number of irreducible components of $\mathbb{C}[\Gamma_0(p^\lambda)\backslash \mathrm{SL}(2,\mathbb{Z})]$ is, by Frobenius reciprocity, bounded to above by the dimension of the subspace of $\Gamma_0(p^\lambda)$-invariant elements in the space $\mathbb{C}[\Gamma_0(p^\lambda)\backslash \mathrm{SL}(2,\mathbb{Z})]$, which in turn equals the number of double cosets $\Gamma_0(p^\lambda)\backslash \mathrm{SL}(2,\mathbb{Z})/\Gamma_0(p^\lambda)$. But the latter is in fact equal to $\lambda + 1$ as may be easily verified by identifying $\Gamma_0(l)\backslash \mathrm{SL}(2,\mathbb{Z})$ with the projective line over $\mathbb{Z}/p^\lambda\mathbb{Z}$ and considering the natural right action $([x:y], A) \mapsto [(x,y)A]$ of $\Gamma_0(l)$ on it.

Note that from the very definition the character ψ_{p^λ} takes on rational values only.

To investigate further our formula for the dimension for $J_{1,m}(l)$ we have now to consider for each prime p the multiplicity

$$I = \langle \psi_{p^\lambda}, (\theta_m^f)_p\, (\theta_n^g)_p\rangle.$$

Assume that p is not a divisor of m. We show that the last multiplicity equals 0 if p divides n.

Indeed, if $p \neq 2$, then $(\theta^f_m)_p = 1$ and hence I is different from 0 only if $\psi_{p^\lambda} = (\theta^g_n)_p$ (since both characters are irreducible). But the latter is possible only if both characters are trivial, i.e. if p does not divide n (and $\lambda = 0$). Namely, if $p|n$ then $(\theta^g_n)_p$ takes on non-rational values whereas ψ_{p^λ} is a rational character as we saw above. To prove the non-rationality of $(\theta^g_n)_p$ note that $T' = ([1,1;0,1],1)$ assumes on Th_m the eigenvalues $\exp(2\pi i\rho^2/4n)$ $(0 \leq \rho < 2n)$. It follows that $T = [1,1;0,1]$ assumes in a representation r with character $(\theta^g_n)_p$ only eigenvalues of the form $\exp(2\pi ih\sigma^2/n_p)$ for certain integers σ and with a suitable h (independent of σ and relatively prime to p). Here n_p is the largest power of p dividing n. In [Sko, Proof of Satz 1.8 (iv)] it was proven that 1 occurs among the σ. But then the traces $(\theta^g_n)_p(T^n)$ cannot be invariant under $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for all n since otherwise the characteristic polynomial $\chi(X) = \det\left(X - r(T)\right)$ would have rational coefficients (note that $X^{\dim r}\chi(1/X) = \exp\left(-\sum_{n\geq 1} \mathrm{tr}\, r(T^n)\, X^n/n\right)$), whence the set of eigenvalues of $r(T)$ would contain simultaneously with $\exp(2\pi ih/n_p)$ all primitive n_p-th roots of unity.

A similar argument works for $p = 2$. Here we have $(\theta^f_m)_2 \neq 1$. However, T' assumes in a representation with character $(\theta^f_m)_2$ only the two eigenvalues 1 and $\exp(2\pi ih/4)$ for some odd h. Moreover, the eigenvalues of T' in a representation of $(\theta^g_n)_2$ are of the form $\exp(2\pi ik\rho^2/4n_2)$ for suitable ρ including $\rho = 1$ and some odd k (independent of ρ). It follows that the set of eigenvalues of T in a representation with character $(\theta^f_m)_2\,(\theta^g_n)_2$ does not contain all primitive $4n_2$-th roots unity if $n_2 > 1$. As above we deduce from this that, for $n_2 > 1$, the traces $(\theta^f_m)_2\,(\theta^g_n)_2(T^n)$ cannot all be rational.

We conclude that $\langle 1_{\Gamma_0(l)}, \mathrm{Res}_\Gamma\, \theta^f_m\, \theta^g_n\rangle = 0$ unless l_2, the part of l which is relatively prime to m, is also relatively prime to n. But if l_2 is relatively prime to n then

$$\langle 1_{\Gamma_0(l)}, \mathrm{Res}_\Gamma\, \theta^f_m\, \theta^g_n\rangle = \langle 1_{\Gamma_0(l_1)}, \mathrm{Res}_\Gamma\, \theta^f_m\, \theta^g_n\rangle.$$

Indeed, let $\phi \in Th^g_n|U_0 \otimes Th^f_m$ be invariant under $\Gamma_0(l)$, and let $A \in \Gamma_0(l_1)$. Choose $A' \in \mathrm{SL}(2,\mathbb{Z})$ such that $A \equiv A' \bmod N$ where N is a common multiple of $4m$, $4n$, l_1 relatively prime to l_2, and such that $A' \in \Gamma_0(l_2)$. Then $\phi|A = \phi|A'$ (since $\Gamma(N)^*$ acts trivially on $Th^0_n|U_0$ and Th_m) and $\phi|A' = \phi$ (since $A' \in \Gamma_0(l_1) \cap \Gamma_0(l_2) = \Gamma_0(l)$). Thus ϕ is invariant under $\Gamma_0(l_1)$.

The theorem follows now immediately. $\square$

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