Modular forms methods for bounding kissing numbers, regulators and covolumes of arithmetic groups

Nils-Peter Skoruppa
Fachbereich Mathematik – Universität Siegen
Walter-Flex-Str. 3 – 57068 Siegen – Germany
http://www.countnumber.de

In this note we would like to show how one can apply rather elementary methods from the theory of modular forms to deduce absolute bounds for various algebro-geometric objects. The elementary methods merely comprise Poisson summation, integration over locally compact groups, some manipulation of higher transcendental functions, and Mellin transforms. No knowledge exceeding fundamental facts from algebraic number theory is assumed. On the other hand the obtained results are surprisingly nontrivial.

The first kind of subject which we would like to apply our methods to are kissing numbers.

Kissing numbers

Recall the definition of the kissing number τ_n of dimension n: τ_n is the maximal number of balls of given radius 1 in Euclidean n-space which can be placed around and touching another one of same radius 1 without any two overlapping.

We shall consider here the lattice kissing number λ_n, which is defined similarly to τ_n, but with the restriction that the centers of balls in question belong to one and the same lattice. More precisely,

$$\lambda_n = \max\{a(L) : L \text{ a lattice in euclidean } n\text{-space}\},$$

where $a(L)$, for a lattice L, is the number of points of L of minimal positive distance to the origin. (Placing a ball of radius $a(L)/2$ around these points and around the origin gives a 'kissing configuration' as described before.) Note that clearly $\lambda_n \le \tau_n$.

The kissing numbers (of both kinds) are known only for a finite number of dimensions: λ_n is known for $n \le 9$ and $n = 24$, whereas τ_n is known for the following values of n only

n	1	2	3	8	24
τ_n	2	6	12	240	196 560

For these five n one has $\tau_n = \lambda_n$, but it is known that $\tau_9 < \lambda_9$. For all this and more details see [Con-Slo], p.23.

For arbitrary n one has the following asymptotic bound ([Kab-Lev]):

$$\tau_n \leq 2^{0.401\,(n(1+o(1))} \leq 1.32043^{n(1+o(1))}.$$

The original proof of Kabatiansky-Levenshtein is quite complicated. But here is a short proof which shows how one can obtain such type of asymptotic bound rather easily, however, only for lattice kissing numbers.

Let L be a lattice in Euclidean n-space. For positive real t set

$$\theta(t) = \theta_L(t) = \sum_{x\in L} \mathrm{e}^{-\pi t x^2} = 1 + a(L) t^{-\pi r t} + \ldots,$$

where r is the smallest non-zero length of points in L. For bounding $a(L)$ we can and will assume, after rescaling L if necessary, that $r = 1$ .he function $\theta(t)$ is obviously *de*creasing in t. On the other hand, $t^{n/2}\theta(t)$ is *in*creasing. Indeed, this follows on using Poisson summation

$$\theta(t)t^{n/2} = \det(L)^{-1}\theta_{L^*}(1/t)$$

(L^* denoting the dual lattice og L.) On taking the derivative we thus find

$$0 \leq \frac{d}{dt}\theta(t)t^{n/2} = \sum_{x\in L} \big(- \pi t x^2 + \frac{n}{2}\big) t^{n/2-1}\mathrm{e}^{-\pi t x^2}.$$

For

$$u := \pi t \geq \frac{n}{2},$$

all terms in the last sum but the first are negative, and on dropping those with $x^2 > 1$ we obtain

$$a(L)(u - \frac{n}{2})\,\mathrm{e}^{-u} \leq \frac{n}{2}.$$

The left hand side takes its maximum for $u = \frac{n}{2} + 1$ (which is in the valid range), and hence we find

$$a(L) \leq \frac{n}{2}\mathrm{e}^{\frac{n}{2}+1}.$$

For example, $a(L) \leq 2, 7, 18$ for $n = 1, 2, 3$. In any case we have proved

Theorem.

$$\lambda_n \leq e^{\frac{n}{2}(1+o(1))} \leq 1.65^{n(1+o(1))}.$$

This is not as sharp as the Kabatiabski-Levenstein bound. However, we shall indicate in a moment how a refinement of the above method could be used to prove a much sharper asymptotic bound. The idea is roughly to replace the exponential function in the definition of θ by another function which reduces the error made when discarding the terms with $x^2 > 1$. For this it seems to be easier to pass to Dirichlet series, i.e. to apply Mellin transformation, since the higher transcendental functions have usually simple Mellin transforms, e.g. ratios of products of Gamma functions with shifted arguments. We shall illustrate these techniques in more precise terms in the next section [1].

We end this section by noting in passing that the above arguments, applied to a modular form, say on the full modular group, yield a quick proof of the following result[2]:
An (elliptic) cusp form on the full modular group has positive as well as negative Fourier coefficients.

Regulators

Possibly the most mysterious and subtle elements of a number fields are the units. They show a rather paradoxical behavior. Sometimes the have a high complexity:

$$K = \mathbb{Q}(\sqrt{611953}), \quad \mathbb{Z}_K^* = \pm\varepsilon^{\mathbb{Z}}, \quad \varepsilon^2 - m\varepsilon - 1,$$

where m is a natural number with 499 digits. (We use $\mathbb{Z}_K$ for the ring of integers of K.) Sometimes they have a small complexity, which can cause problems too: for the height

$$H(\alpha) = \prod_{\alpha' \text{conjugate to } \alpha} \max(1, |\alpha'|)$$

of an algebraic integer it is conjectured that there is an absolute constant $C > 1$ such that

$$H(\alpha) \geq C$$

for all $\alpha \neq 0$ which are not roots of unity (Lehmer conjecture [Lehm]). The numbers for which this conjecture is open are the non reciprocal units i.e. units α which are conjugate to $1/\alpha$, and among these those with very small height occur among the Salem numbers, which are by definition positive real self reciprocal units α whose conjugates, apart from $1/\alpha$, lie on the unit circle $|z| = 1$.

An important measure for a group of units are the regulators. We consider here as an example *relative regulators*: Let L/K be an extension of number fields, and let

$$E = E_{L/K} := \{\varepsilon \in \mathbb{Z}_L^* \,:\, N_{L/K}(\varepsilon) = \text{ root of unity}\},$$

[1]Don Zagier pointed out that the theory of *Ferrar transforms* should provide the correct framework for this cycle of ideas.

[2]We would be grateful for a pointer to the literature where this simple fact (without doubt at least unconsciously known to the reader) is explicitly stated.

be the *group of relative units of this extension.* It is easy to show that the rank of E is

$$s := \mathrm{rank}(E) = \#P_L - \#P_K,$$

where P_K is the set of Archimedean places of K (and similarly for P_L). Let ε_j $(1 \leq j \leq s)$ be a system of fundamental units of L/K, i.e. the $\varepsilon_j T$ form a $\mathbb{Z}$-basis for E/T with T denoting the torsion subgroup of E. For each place v of K pick a place $\widetilde{v}$ of L extending v, and let v_j $(1 \leq j \leq s)$ be the places of L different from all the $\widetilde{v}$. Set $e_v = 1$ or $e_v = 2$ accordingly as v is real or complex. The relative regulator is then defined to be

$$\mathrm{Reg}(L/K) := \left| \det \Big(e_{v_k} \cdot \log |\varepsilon_h|_{v_k} \Big)_{1 \leq h,k \leq s} \right|.$$

Here the absolute values $|\alpha|_v$ associated to v are normalized such that

$$|N_{L/Q}(\alpha)| = \prod_{v \in P_L} |\alpha|_v^{e_v}.$$

If ε is a Salem number, then

$$\mathrm{Reg}\,\big(\mathbb{Q}(\varepsilon)/\mathbb{Q}(\varepsilon + 1/\varepsilon)\big) = \frac{n}{2} \log |\varepsilon|$$

with a suitable positive integer n [Chi]. Moreover, it is easy to prove that

$$\frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)} \geq \mathrm{Reg}(L/K).$$

Thus it is interesting to derive lower bounds for relative regulators.

For the regulator $\mathrm{Reg}(L) = \mathrm{Reg}(L/\mathbb{Q})$ of an arbitrary number field L it is known:

Theorem. *([Zim], 1981)* $\dfrac{\mathrm{Reg}(L)}{w_L} \geq .02 \cdot \exp(0.46\, r_1 + 0.1\, r_2)$

Here r_1 and r_2 denote the real and complex places of K.

Moreover it is conjectured *(Conjecture of Martinet and Bergé): There are absolute constants $C_1 > 0$ and $C_2 > 1$ such that*

$$\mathrm{Reg}(L/K) \geq C_0 C_1^{[L:K]}.$$

One can derive such bounds by generalizing the method which we used before for the kissing numbers. For this consider $E_{\mathbb{R}} = E \otimes \mathbb{R}$ as submodule of $\mathbb{R}_+^{P_L}$ via the embedding

$$E_{\mathbb{R}} = E \otimes \mathbb{R} \to \mathbb{R}_+^{P_L}, \quad \varepsilon \otimes 1 \mapsto \big\{ |\varepsilon|_v \big\}.$$

Here $\mathbb{R}_+$ is the (multiplicative) group of positive real numbers. For $\alpha \in K$ and $x \in \mathbb{R}_+^{P_L}$ let

$$\sigma(\alpha, x) = \sum_{v \in P_L} e_v |\alpha|_v^2 x_v^2.$$

Note that

$$\sigma(\varepsilon\alpha, x) = \sigma(\alpha, (\varepsilon \otimes 1)x)$$

for all $\varepsilon \in E$. Usual Poisson summation gives

$$t^{n/2} \prod_{v \in P_L} x_v^{e_v} \sum_{\alpha \in \mathbb{Z}_L} \exp(-\pi t \sigma(a, x)) = |D_L|^{-1/2} \sum_{\alpha \in \vartheta_L^{-1}} \exp(-\pi t^{-1} \sigma(a, x^{-1}))$$

(where D_L and ϑ_L denote the discriminant and different of L, respectively). Here $x \in \mathbb{R}_+^{P_L}$, $t > 0$, and $n = [L : \mathbb{Q}]$. Note that both sides, as functions of x, are invariant under E. Hence, by integrating over a fundamental domain $E_\mathbb{R}/E$ and the usual trick of "unfolding" the integral, we obtain

$$\begin{aligned} &t^{n/2} \frac{\mu(E_\mathbb{R}/E)}{\# E_{\text{tor}}} + t^{n/2} \sum_{\substack{\alpha \in \mathbb{Z}_L/E \\ \alpha \neq 0}} \int_{E_\mathbb{R}} \exp(-\pi t \sigma(a, x))\, d\mu(x) \\ &= |D_L|^{-1/2} \frac{\mu(E_\mathbb{R}/E)}{\# E_{\text{tor}}} + |D_L|^{-1/2} \sum_{\substack{\alpha \in \vartheta_L^{-1}/E \\ \alpha \neq 0}} \int_{E_\mathbb{R}} \exp(-\pi t^{-1} \sigma(a, x)), d\mu(x). \end{aligned}$$

Here μ denotes any Haar measure on $E_\mathbb{R}$. A natural normalization of μ is the usual Lebesgue measure on E_R, and then one obtains

$$\mu(E_\mathbb{R}/E) = \operatorname{Reg}(L/K).$$

Now one can proceed as with the kissing numbers to derive a lower bound for the relative regulator: the left hand side of the last identity is an increasing function, and taking its derivative gives the inequality

$$\frac{\mu(E_\mathbb{R}/E)}{\# E_{\text{tor}}} \geq \sum_{\substack{\alpha \in \mathbb{Z}_L/E \\ \alpha \neq 0}} \int_{E_\mathbb{R}} \Big(\frac{2u}{n} \sigma(a, x) - 1 \Big) \exp(-u\, \sigma(a, x))\, d\mu(x)$$

for all u $(= \pi t) > 0$. The summands occurring in this inequality are much more complicated than the exponential function and their study is quite cumbersome. Nevertheless, the procedure applied to the kissing numbers before, i.e. dropping all terms with $\alpha \neq 1$ for u large enough and maximizing in u, can be pushed through and gives:

Theorem. *([Fri-Sko 1], 1999) One has*

$$\operatorname{Reg}(L/K) \geq \Big(d_0\, d_1^{[L:K]} \Big)^{[K:\mathbb{Q}]} \quad (d_0 = (0.1/1.15)^{39},\ d_1 = 1.15)$$

As a consequence, the conjecture of Martinet-Bergé holds true at least for $[L:K] \gg 0$.

For $K = \mathbb{Q}$, i.e. for the non relative case, one can obtain, without much effort, a better result by applying Mellin transformation (for $K \neq \mathbb{Q}$ the integrals in our last transformation formula have in general no reasonable Mellin transform which would lead to usual Dirichlet series). For this denote the left hand side of our last Poisson summation formula by

$$u\Big(\frac{\operatorname{Reg}(L)}{\# E_{\mathrm{tor}}} + f(u)\Big),$$

where now $u = t^{n/2}$. It is well-known (this is Hecke's Ansatz for the analytic continuation and the functional equations of partial zeta functions) that, for a suitable choice of μ, one has

$$D^*(s) := \int_0^\infty f(t) t^s \,\frac{dt}{t} = C^s \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} \sum_{\substack{\alpha \in \mathbb{Z}_L \\ \alpha \neq 0}} N_{L/\mathbb{Q}}(\alpha)^{-s} =: C^s \gamma(s) D(s)$$

(for $s > 1$). Here C is a positive constant, whose exact value is not important in the following reasoning. Moreover

$$\operatorname{Res}_{s=0} D^*(s) = -\frac{2^{r_1} \operatorname{Reg}(L)}{w},$$

where w denotes the number of roots of unity in L. Set

$$g(s) := \log s(s-1) D^*(s);$$

then

$$g(0) = \log \frac{2^{r_1} \operatorname{Reg}(L)}{w}.$$

Assume that $g(s)$, for $s > 1$, satisfies the following convexity property:

$$\frac{g(s) - g(0)}{s} \leq g'(s), \text{ i.e. } g(0) \geq g(s) - s g'(s).$$

Unfortunately, we do not know any obvious reason why this should hold true. Nevertheless, this inequalities are indeed correct, as we shall show in a moment. Taking exponentials in the last inequality we obtain

$$\frac{\operatorname{Reg}(L)}{w} \geq \frac{s(s-1)}{2^{r_1}} \gamma(s) D(s) \exp\Big(-1 - \frac{s}{s-1} - s\frac{\gamma'}{\gamma}(s) - s\frac{D'}{D}(s)\Big).$$

Using $D(s) > 1$ and $-\frac{D'}{D}(s) > 0$ we obtain the following lower bounds:

$$\frac{\operatorname{Reg}(L)}{w} \geq \frac{s(s-1)}{2^{r_1}\mathrm{e}} \gamma(s) \exp\Big(- \frac{s}{s-1} - s\frac{\gamma'}{\gamma}(s)\Big).$$

Choosing $s = 4/3$ we find:

Theorem. *([Sko 1], 1993)*

$$\mathrm{Reg}(L) \geq 0.00299 \cdot \exp(0.48\, r_1 + 0.06\, r_2).$$

It remains to prove the convexity property.

Lemma. *Assume that $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies $f \geq 0$, that the Mellin transform $Z(s) := \int_0^\infty f(t) t^s \frac{dt}{t}$ converges for $s > 1$ and that $t(C + f(t))$ is increasing for some constant C. Then*

$$C \geq \frac{s(s-1)}{e} Z(s) \exp\Big(- \frac{s}{s-1} - s \frac{Z'}{Z}(s) \Big).$$

Proof. Set

$$\log_+(t) := \log \max(1, t), \quad w(t) := t \log_+(1/t).$$

Note $w(t) = 0$ for $t > 1$ and $w'(t) = -(1 + \log t)$ for $t < 1$. The function

$$I(a) := \int_0^\infty (C + f(t)) t\, w\big((at)^{s-1}\big) \, \frac{dt}{t}$$

is decreasing in a (set $t/a \mapsto t$) since $t(C + f(t))$ is increasing by assumption. Thus $I'(a) \leq 0$, i.e.

$$\int_0^\infty (C + f(t)) t^s \, w'\big((at)^{s-1}\big) \, \frac{dt}{t}$$

(the reader may verify by himself that the discontinuity of $w'(t)$ at $t = 1$ can indeed be ignored). Inserting the formula for $w'(t)$ then gives

$$C \int_0^{1/a} t^s \big(1 + (s-1) \log(at)\big) \, \frac{dt}{t} \geq - \int_0^{1/a} f(t) t^s \big(1 + (s-1) \log(at)\big) \, \frac{dt}{t} \geq$$
$$- \int_0^\infty f(t) t^s \big(1 + (s-1) \log(at)\big) \, \frac{dt}{t} = -(1 + (s-1) \log a) Z(s) - (s-1) Z'(s).$$

Choosing

$$\log a = -\frac{d}{ds} \log s(s-1) Z(s) = -\frac{1}{s} - \frac{1}{s-1} - \frac{Z'}{Z}(s)$$

gives the desired inequality. □

A variation of the last lemma may be applied to the kissing number problem. For this we replace the exponential function in the definition of the theta series considered in the first section by a suitable Bessel function. More precisely, we start with the identity

$$t \sum_{x \in L} F(t^{1/n} |x|) = c_0 \sum_{x \in L^*} \max\big(0, (1 - t^{-2/n} x^2)^p\big),$$

where

$$F(t) = J_{q-1}(2\pi t)/t^{q-1},\ q = \frac{n}{2} + p + 1.$$

Here we have to assume $q > n + \frac{1}{2}$ (to assure that the series on the left converges absolutely), and c_0 is a positive constant, whose precise value is not important. Writing the left hand side in the form $t(C + f(t))$ we find $C = \pi^{q-1}/\Gamma(q)$ and, for the Mellin transform of $f(t)$, the formula

$$\frac{n}{2}\pi^{q-1-ns}\frac{\Gamma\left(\frac{ns}{2}\right)}{\Gamma\left(q-\frac{ns}{2}\right)}\sum_{\substack{x\in L\\ c\neq 0}}(x^2)^{-s}.$$

Applying the inequality of the preceeding lemma and using in the same manner as before

$$D(s) := \sum_{\substack{x\in L\\ c\neq 0}}(x^2)^{-s} \geq a(L), \quad \frac{D'}{D}(s) < 0,$$

we obtain an estimate to above for $a(L)$.

Unfortunately we cannot apply the preceeding lemma literally, since $f(t)$ is not nonnegative. However, it is still possible to prove a similar lemma for not necessarily nonnegative functions f (and for the price of a slightly weaker inequality). With a suitable choice of q and s we are finally able to deduce:

Theorem. $\lambda_n \leq 1.359^{n(1+o(1))}$.

For a detailed proof the reader is referred to [Sko 2].

We do not know whether the sketched method is suited (with a more clever choice of the function $F(t)$) to deduce an asymptotic bound as sharp as the one of Kabatiansky-Levenshtein or even interesting bounds in specific dimensions.

Covolumes

We conclude this report by speculating about further applications of the ideas sketched so far. The following theorem is known:

Theorem. *(Kajdan-Margoulis) Let G be a semi-simple connected Lie group without compact factors (i.e. without any nontrivial compact connected subgroup), and let μ be a Haar measure on G. Then there is a constant c_μ such that, for all discrete subgroups Γ of G, one has $\mu(G/\Gamma) \geq c_\mu$.*

It should be possible to obtain such bounds, in the case of arithmetic subgroups, by generalizing the ideas of the preceeding sections. The starting point would be as before a suitable Poisson summation type formula involving $\mu(G/\Gamma)$.

To become more precise, let G be a locally compact group and Γ a discrete subgroup. Suppose that G acts on an Euclidean space $(E, \langle\ ,\ \rangle)$ which contains a lattice L stable under the action of Γ. Using Poisson summation and the method of unfolding the integral one would expect a *generalized Poisson summation formula* of the form

$$\mu(G/\Gamma) v f(0) + v \sum_{\substack{x \in L/\Gamma \\ x \neq 0}} \frac{1}{|\Gamma_x|} f(x) = \mu(G/\Gamma) v^* \widetilde{f}(0) + v^* \sum_{\substack{y \in L^*/\Gamma^* \\ y \neq 0}} \frac{1}{|\Gamma^*_y|} \widetilde{f}(y).$$

Here f is a G-invariant function on E, we use L^* for the dual lattice of L with respect to $\langle\ ,\ \rangle$, and $v = \mathrm{vol}(E/L)^{1/2}$, $v^* = \mathrm{vol}(E/L^*)^{1/2}$. Moreover, L^*/Γ^* means the set of orbits with respect to the action of the g^* $(g \in \Gamma)$, where the star indicates dual action on E, i.e. the action defined by $\langle g^* x, y\rangle = \langle x, g^{-1} y\rangle$ for all x, y. Similarly Γ^*_y is the stabilizer of y with respect to the the dual action of Γ. Finally, the function $\widetilde{f}$ is an integral transform which does not depend on Γ or L, but depends only on G and the choice of the Haar measure μ, which also defines the covolume $\mu(G/\Gamma)$.

Suitable choices of f and following the procedure sketched for the bounds for relative regulators, would then yield lower bounds for $\mu(G/\Gamma)$.

It is clear that such a generalized Poisson summation formula will be valid only under suitable hypothesis on f and on the action of G, whose precise formulation needs a careful analysis. The case of the orthogonal group $G = \mathrm{O}(V)$ is well-known and almost trivial: here $\widetilde{f}$ is essentially a Hankel transform. The case where Γ is the group of units relative to an extension of number fields L/K modulo torsion, where $G = \Gamma \otimes \mathbb{R}$, and where L is a fractional ideal of L, considered as sublattice of $L \otimes_{\mathbb{Q}} \mathbb{R}$, is treated in the following theorem (we use the notations introduced in the preceeding section):

Theorem. *([Fri-Sko 2], 2000) Let L/K be an extension of number fields, $\mathfrak{A}$ an ideal of L, let $E := \{\varepsilon \in O_L^* \mid \mathrm{N}_{L/K}(\varepsilon)$ root of unity of $K\}$, and P_K be the set of Archimedean places of K. Then, for all smooth and compactly supported functions $f : \mathbb{R}_+^{P_K} \to \mathbb{C}$, one has*

$$\sum_{\alpha \in \mathfrak{A}/E\ \ \alpha \neq 0} f\Big(\big\{|\mathrm{N}_{L/K}(\alpha)|_v\big\}_{v \in P_K}\Big) = \frac{2^{r_1} (2\pi)^{r_2} \operatorname{Reg}(L/K)}{\sqrt{|D_L|}\, \mathrm{N}_{\mathrm{L}/\mathbb{Q}}(\mathfrak{A}) w_L} \int_{\mathbb{R}_+^{P_K}} f(y)\, dy + \frac{2^{r_2}}{\sqrt{|D_L|}\, \mathrm{N}_{\mathrm{L}/\mathbb{Q}}(\mathfrak{A})} \sum_{\substack{\alpha \in \mathfrak{A}^{-1}\vartheta_L^{-1}/E \\ \alpha \neq 0}} \widetilde{f}\Big(\big\{|\mathrm{N}_{L/K}(\alpha)|_v\big\}_{v \in P_K}\Big),$$

where
$$\widetilde{f}(y) = \pi^{[L:\mathbf{Q}]/2} \int_{\mathbf{R}_+^{P_K}} f(x) \prod_{v \in A_K} k_{p_v,q_v}(x_v y_v)\, dy.$$
Here, for $v \in P_K$ we use p_v, q_v for the number of real and complex places of L extending v, and $k_{p,q}(t) = (\frac{2}{\sqrt{\pi}}\cos(\pi t))^{*p} * J_0(4\pi\sqrt{t})^{*q}$ (the $*$ denotes Mellin convolution).

The assumption about f is too restrictive (in fact, it makes the sum on the left finite) and must be reconsidered for real applications. However, the theorem indicates a direction which may be worthwhile further investigations.

References

[Chi] T. Chinburg, Salem numbers and L-functions, J. Number Theory 18 (1984), 213–214

[Con-Slo] J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, Springer Verlag, 1988.

[Fri-Sko 1] —, *Relative Regulators of number fields*, Invent. Math. 135 (1999), 115–144

[Fri-Sko 2] —, *A Poisson summation formula for extensions of number fields*, J. London Math. Soc. (2) 61 (2000), 36–50

[Kab-Lev] G.A. Kabatiansky and V.I. Levenstein, *Bounds for packings on a sphere and in space*, Problems of Information Transmission 14 (No. 1, 1978), 1–17

[Lehm] D.H. Lehmer, Factorization of certain cyclotomic functions, Ann. Math. 34 (1933), 461–479.

[Sko 1] N-P. Skoruppa, *Quick lower bounds for regulators of number fields*, Enseign. Math. 39 (1993), 137–141

[Sko 2] —, *Quick estimates for kissing numbers*, preprint 1996, submitted.

[S-W] Stein and Weiss, Introduction to Fourier analysis on Euclidean spaces. Princeton University Press 1971

[Zim] R. Zimmert *Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung*, Invent. Math. 62 (1981), 367—380