Examples of $\mathrm{SL}(2,\mathbb{Z})$ – invariant spaces spanned by certain modular units

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Abstract

Characters of rational vertex operator algebras (RVOAs) arising in 2-dim. conformal field theories often belong (after suitable normalisation) to the (multiplicative) semigroup E^+ of modular units whose Fourier expansions are in $q^\alpha(1 + q\,\mathbb{Z}_{\geq 0}[[q]])$ for some rational number α. If even all characters of a RVOA have this property then we have an example of what we call modular sets, i.e. finite subsets of E^+ whose elements (additively) span a vector space which is invariant under the usual action of $\mathrm{SL}(2,\mathbb{Z})$. The classification of modular sets and RVOAs seem to be closely related. In this note we prove a stronger version of a certain inequality which allows to compute several explicit examples of modular sets contained in a natural semi-subgroup E_* of the semi-group E^+ of modular units which have non-negative integer Fourier coefficients.

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1 Introduction

In the present note we report on the results of a systematic search for modular sets, and on possible interpretations of these. We also give an improvement of a certain inequality which is essential for the computation of modular sets.

We recall from [ES05] the definition of modular sets. A modular unit is a holomorphic function $f(z)$ on the Poincaré upper half plane $\mathbb{H}$, without zeroes in $\mathbb{H}$, and such that $f(z)$ is invariant under some congruence subgroup of $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ (i.e. $f(Az) = f(z)$ for all A in some congruence subgroup, where $z \mapsto Az$ is the usual Moebius transformation associated to the matrix A). We set

$$E = \big\{ f \text{ modular unit} \mid f \in q^r(1 + q\mathbb{Z}[[q]]) \text{ for some } r \in \mathbb{Q} \big\},$$

where we use q^r for the function $q^r(z) = \mathrm{e}^{2\pi i r z}$. A modular set is a nonempty finite subset of E whose elements all have nonnegative Fourier coefficients, and such that the complex vector space of functions on $\mathbb{H}$ spanned by its elements is invariant under Γ.

In [ES96] it was pointed out that the sets of conformal characters of various 2-dimensional rational conformal field theories (RCFT) are modular (after suitable normalisation of the conformal characters). Moreover, as explained in [ES05], modular sets arise in the context of Ramanujan-Rogers type identities. The connection between these two phenomena, however, seems to be completely unclear.

It seems thus to be reasonable to do some computer aided research for modular sets. As starting point we take the more theoretical results of [ES05]: Obviously, modular units form a group with respect to usual multiplication of functions, and it contains E as a subgroup. As shown in [ES05] the group E is generated by the special modular units

$$[r]_l = q^{-l\mathbb{B}_2\left(\frac{r}{l}\right)/2} \prod_{\substack{n \equiv r \bmod l \\ n>0}} (1-q^n)^{-1} \prod_{\substack{n \equiv -r \bmod l \\ n>0}} (1-q^n)^{-1},$$

where $l \geq 1$ and r run through all postive integers such that l does not divide r, and where $\mathbb{B}_2(x) = y^2 - y + \frac{1}{6}$ with $y = x - \lfloor x \rfloor$ as the fractional part of x. The semisubgroup of units in E with nonnegative coefficients contains the semigroup E_* generated by all the $[r]_l$. Since among all modular sets which we know the nontrivial ones always lie entirely in E_* it is reasonable to restrict our research to those.

For this we put, for integers $l, n \geq 1$,

$$E_n(l) = \Big\{ [r_1, \dots, r_k]_l := \prod_{j=1}^{k} [r_j]_l \mid k \leq n \text{ and } r_j \in \mathbb{Z},\ l \nmid r_j \Big\}.$$

Using the obvious distribution relations

$$[r]_l = \prod_{\substack{s \bmod m \\ s \equiv r \bmod l}} [s]_m \qquad (l, m \geq 1,\ l|m),$$

it is clear that any modular set is contained in $E_n(l)$ for suitable n, l. Since the union of modular sets is modular we may speak of the unique maximal modular subset $S_n(l)$ of $E_n(l)$. For later convenience we define $E_*(l) = \cup_{n\geq 0} E_n(l)$ and $E_n(*) = \cup_{l>1} E_n(l)$ and similarly $S_*(l) = \cup_{n\geq 0} S_n(l)$, $S_n(*) = \cup_{l>1} S_n(l)$.

It is possible to determine $S_n(*)$ for small values n explicitly. In [ES05] this has been done for $n = 1, 2, 3$. Here we extend the results of *loc cit.* to all $n \leq 11$. Our explicit results are contained in the tables of section 4. Before we recall how the sets $S_n(*)$ can be computed we would also like to mention some results about the sets $S_*(l)$. In this case we only know $S_*(l)$ completely for $l = 2, 3, 4, 5, 7$: for $l \leq 4$ one has $S_*(l) = \emptyset$, for $l = 5$ one has $S_*(5) = E_*(5)$ and for $l = 7$ one finds that $S_*(7)$ is 'generated' by only two modular sets (see section 3 for details).

Finally, let us recall how, for fixed n, one can compute $S_n(*)$.

By Theorem 2 of [ES05] we know that if $E_n(l)$ contains a modular subset (different from the trivial modular set $\{1\}$) then $l \leq 13.7^n$. Actually we will show in section 2 that this estimate can be sharpened to the much stronger estimate $l \leq 5n$ (*cf.* the Corollary in section 2). This means that in order to determine $S_n(*)$ for fixed n we only have to consider the finitely many l-values $2, \ldots, 5n$.

Assume form now on that n and l are fixed (and $l \leq 5n$). To compute $S_n(l)$ we recall from Lemma 3.3 of [ES05] that, for all divisors t of l, one has for each $\pi = [a_1, \ldots, a_k]_l \in S_n(l)$ $(k \leq n)$

$$\max_{a \bmod {}^* t} \sum_{j=1}^{k} \mathbb{B}_2\big(\frac{a_j a}{t}\big) \leq \frac{n}{6t^2}.$$

Furthermore, for each divisor t there exists at least one $\pi \in S_n(l)$ such that equality holds. The asterisk indicates that a runs through a complete set of representatives for the primitive residue classes modulo t. We call a subset $T \subset E_n(l)$ pre-modular if it satisfies this conditions.

As for given n and l the set $E_n(l)$ contains only finitely many elements we can use a computer programme to determine those of its elements that are contained in the maximal pre-modular set $T \subset E_n(l)$. If it is not clear by other means whether T actually is modular, then one can proceed as follows.

Let T_1 be the set of all $\pi \in T$ such that $\pi(-1/z)$ is a linear combination of the functions in T. Note that by standard arguments from the theory of modular forms it suffices to check for $N = N(l,n)$ many Fourier coefficients only, where $N(l,n)$ depends on l and n, and can be determined explicitly. The Fourier coefficients of $\pi(-1/z)$ can be read off from the corollary to Theorem 5.1 of *loc cit.*. Similarly, construct T_2 from T_1, T_3 from T_2 and so forth. Either some T_k is empty, and then T contains no modular set, or $T_k = T_{k+1} \neq \emptyset$ for some k, and then T_k is the maximal modular set in T and therefore in $E_n(l)$.

Finally, note that by a simple argument given in section 4 of *loc cit.* one can show that for $n < 12$ any modular set in $E_n(l)$ can be decomposed into a disjoint union of modular sets each of which is contained in $E_k(l) \setminus E_{k-1}(l)$ for some $k \leq n$, *i.e.* where all products have the same 'length'. For $n \geq 12$, however, this is no longer true (see the example in the conclusion); we do not know whether the corresponding statement for maximal modular sets holds even for $n \geq 12$.

The rest of this short note is organised as follows: Section 2 contains the prove of the estimate $l \leq 5n$. In section 3 we have collected a few facts about the sets $S_*(l)$ for small l and in section 4 we give a complete list of all modular subsets $S_n(*)$ for $n \leq 11$. We conclude with section 5 with a few remarks and open questions.

2 The estimate $l \leq 5n$

The group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ is acting on the field of all meromorphic functions on the Poincaré upper half plane $\mathbb{H}$ by $(f, A) \mapsto f \circ A$, where we use $(f \circ A)(z) = f(Az)$ with $z \mapsto Az$ denoting the Moebius transformation associated to the matrix A.

We are interested in the Γ-invariant subfield $\mathrm{Mod}(\mathbb{H})$ of all modular functions. By the term modular function we understand a meromorphic function $f(z)$ on the upper half plane $\mathbb{H}$ which is invariant under some congruence subgroup of Γ. If f is a modular function and $s \in \mathbb{P}(\mathbb{Q})$, say $s = A\infty$ for some $A \in \Gamma$, then $f \circ A$ possesses a Laurent development in powers of $q^{1/N}$ with a suitable positive integer N. Here we use q^r for the function $q^r(z) = \mathrm{e}^{2\pi i r z}$. If q^α is the smallest power occurring in this Laurent series we set

$$\mathrm{ord}_s(f) := \alpha.$$

The order of f in α is clearly independent of the choice of A.

Finally, for a finite dimensional Γ-submodule X of $\mathrm{Mod}(\mathbb{H})$ we put

$$\nu_X := \min_{f \in X} \mathrm{ord}_s(f),$$

where s is any cusp. It is easy to see that ν_X does not depend on the particular choice of s. In this section we prove

Theorem. *Let X be a finite dimensional Γ-module of nonconstant modular functions. Then $\nu_X \leq -\frac{1}{60}$.*

Since, for a modular set X we have $\nu_X = -\frac{n}{12l}$ [ES05], we obtain as immediate consequence

Corollary. *Let $X \neq \{1\}$ be a modular set in $E_n(l)$. Then $l \leq 5n$.*

Proof of the theorem. By η we denote the Dedekind eta function, respectively, i.e. we use

$$\eta = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

It is well known that $\kappa(A, z) := \eta(Az)/\eta(z)$ is a multiplier system of weight $\frac{1}{2}$ for Γ (i.e., for any $A \in \Gamma$ one has $\kappa(A, z)^2 = cx + d$, where (c, d) is the lower row of A, and, obviously from the definition, one has $\kappa(AB, z) = \kappa(A, Bz)\kappa(B, z)$ for all $A, B \in \Gamma$). In particular Γ acts on meromorphic functions f on the upper half plane by

$$(f|A)(z) = f(Az)\kappa(A, z)^{-1}.$$

By a modular function of weight $\frac{1}{2}$ we understand a holomorphic function f on $\mathbb{H}$ such that f/η is a modular function (hence in particular invariant under some *congruence* subgroup) whose order in each cusp is $\geq -\frac{1}{24}$. (It is easily checked that the latter is equivalent to the usual condition of being "holomorphic in the cusps") Clearly, Γ acts on the space $M_{1/2}$ of all modular forms of weight $\frac{1}{2}$ by $(f, A) \mapsto f|A$.

We may assume $\nu_X \geq -\frac{1}{24}$, since otherwise there is nothing to prove. But then $\eta \times X$ is a Γ-submodule of $M_{1/2}$. Moreover, we may obviously restrict to the case of an irreducible X. Then $\eta \times X$ is irreducible too. Thus, it suffices to prove that, for any irreducible Γ-submodule Y of $M_{1/2}$, we have

$$\min_{f \in Y} \operatorname{ord}_\infty(f) \leq \frac{1}{24} - \frac{1}{60} = \frac{1}{40}.$$

By a theorem of Serre and Stark [SS77] one knows that $M_{1/2}$ is the sum of the spaces

$$T_m := \operatorname{span}\big\{\theta_{m,\rho} := \sum_{r \in \rho + 2m\mathbb{Z}} q^{r^2/4m} \mid 0 \leq \rho \leq m\big\},$$

where m runs through all positive integers. It is a well-known fact that T_m is a Γ-submodule of $M_{1/2}$. The decomposition of T_m in irreducible submodules was given in [S85], from which the complete decomposition of $M_{1/2}$ was then also deduced. We explain this decomposition in more detail.

If m and n are positive integers such that m/n is a perfect square, say $m = nd^2$, then T_n is a submodule of T_m. Indeed,

$$n\theta_{n,\rho} = \sum_{x \equiv \rho d \bmod 2nd} q^{x^2/4m} = \sum_{\substack{\sigma \bmod 2m \\ \sigma \equiv \rho d \bmod 2m/d}} \theta_{m,\sigma}.$$

The hermitian scalar product on T_m defined by $\langle \theta_{m,\rho}, \theta_{m,\sigma} = 1$ if $\rho \equiv \sigma \bmod 2m$, and $=0$ otherwise, is invariant under the action of Γ [S85] (p.11). Thus, if we let T_m^0 be the orthogonal complement of the sum of all T_n with m/n a perfect square and $n < m$, then T_m^0 is Γ-invariant.

Next let

$$G(m) = \{\varepsilon \in \mathbb{Z}/2m\mathbb{Z} \,|\, \varepsilon^2 \equiv 1 \bmod 4m\}.$$

If χ is a character of $G(m)$, we let

$$T_m^0(\chi) := \Big\{ \sum_{\rho \bmod 2m} \psi(\rho)\, \theta_{m,\rho} \in T_m^0 \mid \forall \varepsilon \in G(m), \rho : \psi(\varepsilon\rho) = \chi(\varepsilon)\psi(\rho) \Big\}$$

Clearly, $T_m^0(\chi) = 0$ if χ is odd. However, for even χ, it was shown in [S85] (Satz 1.8, p.22), that $T_m^0(\chi)$ is Γ-invariant and irreducible, and that the $T_m^0(\chi)$ ($m \geq 1$, $\chi \in \widehat{G(m)}$ even) are pairwise nonequivalent as Γ-modules. Moreover, [S85] (Satz 5.2, p.101),

$$M_{1/2} = \bigoplus_{m \geq 1} \bigoplus_{\substack{\chi \in \widehat{G(m)} \\ \chi \text{ even}}} T_m^0(\chi).$$

Hence, any irreducible submodule Y of $M_{1/2}$ equals $T_m^0(\chi)$ for some m and (even) χ. The theorem now follows from the fact that, for $Y = T_m^0(\chi)$, one has

$$\nu_Y = \begin{cases} 0 & \text{if } \chi = 1, \\ \frac{1}{4m} & \text{if } \chi \neq 1, \end{cases}$$

as we shall show in a moment. Indeed, this identity implies $\nu_Y \leq \frac{1}{40}$ if $m \geq 10$ or if $\chi = 1$. But, for $m < 10$ one has $G(1) = 1$, $G(m) = \{\pm 1\}$ for $m = 2, 3, 4, 5, 7, 8, 9$, and $G(6) = \{\pm 1| \times \{\pm 1\}$. Hence, for $m < 10$, an even $\chi \neq 1$ exist only if $m = 6$. But then $T_6^0(\chi)$ is spanned by η [S85] (p.26), and hence Y/η equals the submodule of constant functions, which is excluded by the hypothesis of the theorem.

The above identity can be proved as follows. For any function ψ from $\mathbb{Z}/2m\mathbb{Z}$ to $\mathbb{C}$ define Θ_ψ by

$$\Theta_\psi = \sum_{\rho \bmod 2m} \psi(\rho)\, \theta_{m,\rho}.$$

Firstly, consider the case of $Y = T_m^0(\chi)$ for $\chi \neq 1$. Assume that Θ_ψ is an element of $T_m^0(\chi)$. Then we know by the definition of $T_m^0(\chi)$ that $\psi(\varepsilon\rho) = \chi(\varepsilon)\psi(\rho)$ for all $\varepsilon \in G(m)$ and $\rho \bmod 2m$ so that for $\rho = 0$ we find $\psi(0) = \chi(\varepsilon)\psi(0)$. As $\chi \neq 1$ this implies $\psi(0) = 0$ and, hence, $\nu_Y > 0$. It remains to show that $\nu_Y = \frac{1}{4m}$ for $\chi \neq 1$. Note that $\nu_Y = \frac{1}{4m}$ is the smallest possible for ν_Y strictly larger than zero. The equality $\nu_Y = \frac{1}{4m}$ is now easyly obtained from the fact that Θ_ψ with ψ given by

$$\psi(\rho) = \begin{cases} \chi(\rho) & \text{if } \rho \in G(m), \\ 0 & \text{else,} \end{cases}$$

is contained in $T_m^0(\chi)$ (obviously ψ satisfies $\psi(\varepsilon\rho) = \chi(\varepsilon)\psi(\rho)$ for all $\varepsilon \in G(m)$ and $\rho \bmod 2m$; the orthogonality of Θ_ψ to all $\theta_{n,\rho}$ with $m = d^2 n$ is eaily checked).

Secondly, we show that $\nu_Y = 0$ for $Y = T_m^0(1)$. Here we consider Θ_ψ with ψ defined by

$$\psi(\rho) = \sum_{\varepsilon \in G(m)} \exp(\frac{2\pi i \varepsilon\rho}{2m}).$$

Again, it is obvious that $\psi(\varepsilon\rho) = \psi(\rho)$ for all $\varepsilon \in G(m)$ and $\rho \bmod 2m$ and the orthogonality property is a simple exercise.

$\square$

3 Modular subsets of $E_*(l)$ with $l = 2, 3, 4, 5, 7$

The maximal modular subsets of $E_*(l)$ with $l = 5, 7$ can easily be described explicitly and for $l = 2, 3, 4$ there do not exist any modular subsets of $E_*(l)$ at all.

Firstly, note that there are no modular sets in $E_n(l)$ for $l = 2, 3, 4$ and arbitrary n. As only $\pi = [1]_l^n$ is possible for $l = 2, 3$ we find that $\mathrm{ord}_\infty(\pi) = \frac{n}{12} \neq -\frac{n}{12l}$ for $l = 2, 3$. For $l = 4$ we must have $\pi = [1]_4^a[2]_4^b$ $(a + b = n)$ so that $\mathrm{ord}_\infty(\pi) = \frac{a+4b}{24} \neq -\frac{n}{12l} = -\frac{a+b}{48}$ unless $a = b = 0$. Hence $S_n(2) = S_n(3) = S_n(4) = \emptyset$.

Secondly, for $l = 5$ we know that the set $A_1(5) = \{[1]_5, [2]_5\} \subset E_1(5)$ is modular. Hence, for every fixed integer n, the set $\{[1]_5^r[2]_5^s | r + s = n\}$ is also modular and, therefore, $E_n(5)$ itself is modular so that $S_n(5) = E_n(5)$. Note, however, that $\mathbb{Q}[[1]_5, [2]_5]$ is not freely generated as one has e.g.

$$1 + [1]_5[2]_5^{11} + 11[1]_5^6[2]_5^6 = [1]_5^{11}[2]_5.$$

Finally, for $l = 7$ we know from [ES96, ES05] that the sets

$$\begin{aligned} A_1(7) &= \{[1,2]_7, [2,3]_7, [1,3]_7\} \subset E_2(7), \\ A_2(7) &= Y(7) = \{[1,2,3]_7, [1,2,2]_7, [1,1,3]_7, [2,3,3]_7\} \subset E_3(7) \end{aligned}$$

are modular. We show that the maximal modular subset of $E_n(7)$ is generated by these two modular sets. Assume that $\pi = [1]_7^a[2]_7^b[3]_7^c$ $(a+b+c = n)$ is contained in the maximal modular subset of $E_n(7)$. Without loss of generality we can assume that $a = \min(a, b, c)$ (otherwise we can find elements in $\mathrm{SL}(2,\mathbb{Z})$ that map π to $[\lambda]_7^a[2\lambda]_7^b[3\lambda]_7^c$ with $\lambda = 2, 3$). Using Lemma 3.3 of [ES05] implies the following inequalities

$$a \leq b + 2c, \qquad b \leq c + 2a, \qquad c \leq a + 2b.$$

But then π can be written as a product of elements in $A_1(7)$ and $A_2(7)$, namely as

$$\pi = \begin{cases} [1,3]_7{}^{c-b}[1,2,3]_7^{a+b-c}[2,3]_7^{c-a} & \text{if } a \leq b \leq c \leq a+b \\ [1,3]_7{}^{a}[2,3]_7^{2b+a-c}[2,3,3]_7^{c-a-b} & \text{if } a \leq b \leq c \geq a+b \\ [1,2]_7{}^{b-c}[1,2,3]_7^{a+c-b}[2,3]_7^{b-a} & \text{if } a \leq c \leq b \leq a+c \\ [1,2]_7{}^{2a+c-b}[1,2,2]_7^{b-a-c}[2,3]_7^{c} & \text{if } a \leq c \leq b \geq a+c. \end{cases}$$

This shows that the maximal modular subset $S_n(7)$ of $E_n(7)$ is generated by $A_1(7)$ and $A_2(7)$, *i.e.* that

$$S_n(7) = \sum_{2a+3b=n} A_1(7)^a A_2(7)^b.$$

Note, however, that the sum in the last equation is not a direct sum. In particular, the ring (over $\mathbb{Q}$) generated by the functions $[1]_7$,$[2]_7$ and $[3]_7$ is not free as exist relations. A particularly simple one is *e.g.* given by

$$[1,1,2,2,2]_7 + [2,2,3,3,3]_7 = [1,1,1,3,3]_7.$$

4 Tables of modular sets in $E_n(*)$ for $n \leq 11$

In this section we present the results obtained by using the algorithm described in section 1 taking into account the improved bound $l \leq 5n$ proved in section 2.

Table 1: Modular subsets of $E_n(l) \setminus E_{n-1}(l)$ with $l \neq 5, 7$ and $4 \leq n \leq 8$.

$l \backslash n$	4	5	6	7	8
8	$A_2(8)$	–	$N(8)$	–	$A_2(8)^2$
9	–	–	$A_1(9)^2 + A_3(9)$	–	–
10		–	$A_1(5)^3 + A_2(10)$	–	$A_1(5)^4 + A_1(5)A_2(10)$
11	$A_1(11)$	$A^s_{1,3}(11)$	$B_2(11)$	$A_2(11)$	$A_1(11)^2 + D_4(11)$
12	–	–	$T(12)$	–	–
13	–	$A_1(13)$	$A^s_{1,3}(13)$	–	$B_2(13)$
15	–	–	$A_1(15)$	–	–
16	$A_{1,3}(16)$	–	–	–	$A_{1,3}(16)^2 + A_2(8) + T(16)$
17	–	–	–	$A_1(17)$	$A^s_{1,3}(17)$
19	–	–	–	–	$A_1(19)$
20	–	–	$A_{1,3}(20)$	–	–

Table 2: Modular subsets of $E_n(l) \setminus E_{n-1}(l)$ with $l \neq 5, 7$ and $9 \leq n \leq 11$.

$l \backslash n$	9	10	11
8	–	$A_2(8)N(8)$	–
9	$A_1(9)^3 + A_1(9)A_3(9)$	–	–
10	–	$A_1(5)^5 + A_1(5)^2A_2(10)$	–
11	$A_1(11)A^s_{1,3}(11) + A_3(11)$	$A_1(11)B_2(11) + A^s_{1,3}(11)^2$	$A_1(11)A_2(11) + A^s_{1,3}(11)B_2(11)$
13	$A_2(13) + B_3(13)$	$A_1(13)^2 + D_5(13)$	$A_1(13)A^s_{1,3}(13)$
14	–	$A_1(7)A_2(7) + A_2(14)$	–
15	$A_1(5)^3 + A_1(5)A_1(15)$	–	–
17	–	$G_2(17)$	–
19	$A^s_{1,3}(19)$	–	–
20	–	$A_1(5)A_{1,3}(20)$	–
21	$A_1(7) + A_1(21)$	–	–
23	–	$A_1(23)$	$A^s_{1,3}(23)$
25	–	–	$A_1(25)$
28	–	$A_{1,3}(28)$	–

In Table 1 and 2 we list certain modular subsets of $E_n(l) \setminus E_{n-1}(l)$ with $l \neq 5, 7$ and $4 \leq n \leq 11$ (the corresponding sets for $n = 1, 2, 3$ have been given in section 1 of [ES05]). The $\mathrm{SL}(2, \mathbb{Z})$ modules spanned by these modular sets are equal to the $\mathrm{SL}(2, \mathbb{Z})$ modules spanned by the corresponding maximal modular sets. The corresponding maximal modular sets are obtained from sets in Table 1 by adding the additional elements listed in Table 2.

To make the information contained in the tables more compact we will, for S a maximal modular set in $E_n(l) \setminus E_{n-1}(l)$, use $\mathbb{P}(S)$ for the orbits of the $(\mathbb{Z}/l\mathbb{Z})^*$ action on $\{[\bar{a}_1, \ldots \bar{a}_n] \mid [a_1, \ldots, a_n]_l \in S\}$ where the bar denotes reduction modulo l. Note that, if S is a maximal modular set then $\mathbb{P}(S)$

determines S uniquely.

In Table 1 and Table 2 we have used $A_1(k)$, $A_{1,3}(4k)$ and $A^s_{1,3}(l)$ for the modular sets given by the characters of the Virasoro minimal model with $c=c(2,k)$ for odd k, of the Virasoro minimal model with $c=c(3,k)$, $(3,k)=1$ (*cf.* §3.2 of [ES96]) and of the rational models of $W(2,\frac{l-1}{2})$ at $c=c(2l,3)$, $(l,6)=1$ (*cf.* §3.3 of [ES96]), respectively.

Table 3: Additional elements in $\mathbb{P}(S)$ missing in Table 1 and Table 2.

$l\backslash n$	6	8	9	10	11
8	–	$[1,1,1,2,2,2,2,3]_8$	–		–
9	–	–	$[1,1,1,1,2,2,4,4,4]_9$	–	–
10	$[1,1,3,4,4,4]$	$[1,1,1,3,4,4,4,4]_{10}$ $[1,1,2,2,2,3,3,4]_{10}$	– –	$[1,1,1,1,2,3,3,3,5,5]_{10}$, $[1,1,1,2,2,2,2,4,5,5]_{10}$, $[1,1,1,2,2,3,3,4,5,5]_{10}$, $[1,1,2,2,2,2,2,3,3,5]_{10}$, $[1,1,2,2,2,3,4,4,5,5]_{10}$	–
11	–	–	–	$[1,1,1,1,3,3,3,4,5,5]_{11}$, $[1,1,1,2,2,2,4,4,5,5]_{11}$	$[1,1,1,1,2,2,4,4,4,5,5]_{11}$, $[1,1,1,1,2,3,3,3,5,5,5]_{11}$, $[1,1,1,1,2,3,3,4,4,4,5]_{11}$
13	–	–	–	–	$[1,1,1,2,2,3,4,5,5,5,6]_{13}$, $[1,1,1,2,3,3,4,5,5,5,6]_{13}$
14	–	–	–	$[1,1,1,3,4,4,5,6,6,6]_{14}$	–
20	–	–	–	$[1,2,3,4,5,6,7,8,9,10]_{20}$, $[1,2,2,3,5,6,6,7,9,10]_{20}$	–

Furthermore, we have denoted by $L_n(k)$ ($L=A,\ldots,G$, $n\geq 2$) the modular sets associated to

$$
\begin{aligned}
\mathbb{P}(A_2(7)) &= \{[1,2,3]_7,[1,1,3]_7\},\\
\mathbb{P}(A_2(8)) &= \{[1,1,2,4]_8,[1,1,3,3]_8,[1,2,2,3]_8,[1,2,3,4]_8\},\\
\mathbb{P}(A_2(10)) &= \{[1,1,2,2,4,5]_{10},[1,1,2,3,3,5]_{10},[1,1,2,3,4,4]_{10},[1,2,2,3,4,5]_{10}\},\\
\mathbb{P}(A_2(11)) &= \{[1,1,2,2,3,5,5]_{11},[1,1,2,3,3,4,5]_{11}\},\\
\mathbb{P}(A_2(13)) &= \{[1,1,2,2,3,3,5,6,6]_{13},[1,1,2,2,3,4,5,5,6]_{13},[1,1,2,3,3,4,4,5,6]_{13}\},\\
\mathbb{P}(A_2(14)) &= \{[1,1,2,2,3,3,4,6,6,7]_{14},[1,1,2,2,3,3,5,5,6,7]_{14},[1,1,2,2,3,4,4,5,6,7]_{14},[1,1,2,2,3,4,5,5,6,6]_{14}\\
&\qquad [1,1,2,3,3,4,4,4,5,6]_{14},[1,1,2,3,3,4,5,5,6,7]_{14},[1,1,2,3,4,4,5,6,6,7]_{14}\},\\
\mathbb{P}(B_2(11)) &= \{[1,1,2,3,4,5]_{11},[1,1,2,4,4,5]_{11}\},\\
\mathbb{P}(B_2(13)) &= \{[1,1,2,2,4,5,5,6]_{13},[1,1,2,3,3,4,5,6]_{13},[1,1,2,3,4,5,5,6]_{13}\},\\
\mathbb{P}(G_2(17)) &= \{[1,1,2,3,4,4,6,6,7,8]_{17},[1,1,2,3,4,5,5,6,7,8]_{17}\},\\
\mathbb{P}(A_3(9)) &= \{[1,1,1,3,3,4]_9,[1,1,2,2,3,4]_9,[1,1,2,2,4,4]_9,[1,1,2,3,3,4]_9\},
\end{aligned}
$$

$$\begin{aligned}
\mathbb{P}(A_3(11)) &= \{[1,1,1,2,2,4,4,4,5]_{11},[1,1,1,2,3,3,4,4,5]_{11},[1,1,1,2,3,3,4,5,5]_{11},[1,1,2,2,3,3,4,4,5]_{11}\},\\
\mathbb{P}(B_3(13)) &= \{[1,1,1,3,3,4,5,5,6]_{13},[1,1,2,2,3,4,4,5,6]_{13},[1,1,2,2,3,4,5,5,6]_{13},[1,1,2,3,3,4,4,5,6]_{13}\},\\
\mathbb{P}(D_4(11)) &= \{[1,1,1,2,4,4,4,5]_{11},[1,1,2,2,3,4,4,5]_{11}\},\\
\mathbb{P}(D_5(13)) &= \{[1,1,1,2,3,4,4,4,6,6]_{13},[1,1,1,2,3,4,5,5,5,6]_{13},[1,1,2,2,3,3,4,4,5,6]_{13},[1,1,2,2,3,3,4,5,6,6]_{13}\}.
\end{aligned}$$

We have checked that, for $n = 2$, these modular sets equal the sets of characters of the minimal model of WL_n at $c = c(\hat{h}, k)$ with L_n a simple Lie algebra and $\hat{h}$ its dual Coxeter number (*cf.* the conjecture in §4 of [ES96]). For $n > 2$, however we have only checked that the vanishing orders of the modular units in the corresponding modular sets agree with the ones predicted by the conjecture in in *loc. cit.*.

Finally, there are three modular sets which do not seem to be related to any RCFTs

$$\begin{aligned}
\mathbb{P}(N(8)) &= \{[1,1,1,2,3,3]_8,[1,1,1,3,3,4]_8,[1,1,2,2,2,3]_8,[1,1,2,2,3,4]_8\},\\
\mathbb{P}(T(12)) &= \{[1,1,3,3,4,5]_{12},[1,1,3,4,5,5]_{12},[1,2,2,3,4,5]_{12},[1,2,2,3,5,6]_{12},[1,2,3,4,5,6]_{12}\},\\
\mathbb{P}(T(16)) &= \{[1,1,3,3,5,5,7,7]_{16},[1,1,3,4,5,6,7,7]_{16},[1,2,2,3,4,5,7,8]_{16},\\
&\quad [1,2,2,3,5,6,6,7]_{16},[1,2,3,4,4,5,6,7]_{16},[1,2,3,4,5,6,7,8]_{16}\}.
\end{aligned}$$

Note, however, that with

$$\begin{aligned}
T_r &= \{\frac{(\theta_{0,r}-\theta_{0,r+2})}{2\eta},\frac{(\theta_{0,r}+\theta_{0,r+2})}{2\eta},\frac{\theta_{1,r}}{\eta},\ldots,\frac{\theta_{r-1,r}}{\eta},\frac{\theta_{r,r}}{2\eta},\\
&\quad \frac{\theta_{1,r+2}}{\eta},\ldots,\frac{\theta_{r+1,r+2}}{\eta}\frac{\theta_{r+2,r+2}}{2\eta}\}
\end{aligned}$$

one has $T(12) = T_1$ and $T(16) = T_2$. Furthermore, for all but two of the elements π of T_i one has that $\frac{\eta((\tau+1)/2)}{\eta(\tau)}\pi(\tau)$ is a character of a rational superconformal field theory [F93, Section 6].

5 Conclusions and open questions

A few obvious questions remain:

1. The rings $\mathbb{Q}[[r]_l]_{r=1,\ldots\lfloor l/2\rfloor}$ are not freely generated for $l \geq 4$ as they are contained in a function field of a compact Riemann surface and thus

in a field of transcendental degree 1. But how do the relations in these rings look like? We know only a few examples of such relations for small values of l.

2. Is it true that maximal modular sets in $E_n(l)$ can always be written as disjoint unions of modular sets each of which is contained in some $E_k(l) \setminus E_{k-1}(l)$ ($k \leq n$)? So far we know that this is even true for all modular sets in $E_n(l)$ with $n < 12$. For $n \geq 12$ this property does not hold for all modular sets; an counter example is e.g. given by the modular set $E_{12}(5) \setminus \{[1]_5^{11}[2]_5\}$ (*cf.* section 3).

3. Is it true that the sets $S_*(l)$ are generated by finitely many modular sets? We know from section 3 that this is at least true for $l = 5, 7$.

4. Do the modular units in modular sets admit sum formulas similarly to Ramanujan-Rogers type identities? We only know that this is the case for the modular sets $A_1(l)$ and $A_{1,3}(l)$ (see *e.g.* [BMS96]).

5. Are all modular sets related to sets of characters of rational vertex operator algebras? If yes, how can one explain the three exceptional modular sets $T(8)$, $T(12)$ and $N(8)$ in section 4?

6. From the explicit results in section 4 it seems that all (but the three just mentioned exceptional cases) of the maximal modular sets can be interpreted using the (partially conjectured) modular sets described in [ES96, §3+§4]. This indicates that there might be a classification of (maximal) modular sets in terms of some data only related to simple Lie algebras.

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