Quick asymptotic upper bounds for lattice kissing numbers

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Abstract

General upper bounds for lattice kissing numbers are derived using Hurwitz zeta functions and new inequalities for Mellin transforms.

1 Statement of results

Let τ_n be the kissing number in dimension n, i.e. the maximal number of balls of equal size in Euclidean space of dimension n which can touch another one of the same radius without any two overlapping. Similarly let λ_n be the maximal lattice kissing number in dimension n, which is defined like τ_n, but with the restriction that all balls are centred at the points of a lattice and that the centres of the kissing balls have minimal distance to the centre of the kissed one. Alternatively, λ_n is the maximal number of minimal vectors which a lattice L in $\mathbb{R}^n$ can have. The precise values of λ_n and τ_n are known only for finitely many values of n [C-S]. Note that $\lambda_n \leq \tau_n$. The first time when this inequality is strict occurs for $n = 9$. Concerning the asymptotic behaviour of τ_n one knows $\tau_n \geq (1.15470\dots)^{n(1+0(1))}$ [W], and one has the following asymptotic estimate to above of Kabatiansky-Levenshtein [K-L]

$$\tau_n \leq 2^{0.401n(1+o(1))} = (1.32042\dots)^{n(1+o(1))}.$$

As general reference for these and more informations on kissing numbers we refer to [C-S] or [Z].

In the present note we shall prove a general upper bound for λ_n, which we now describe. Assume

1. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a nonzero decreasing and continuous function.

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2. $f(t)$ and the function $F(t)$ defined by $F(|y|) := \int_{\mathbb{R}^n} f(|x|)\, \mathrm{e}^{-2\pi i x y}\, dx$ are $O\big(t^{-(n+\varepsilon)}\big)$ for some $\varepsilon > 0$ as t tends to infinity. Here xy denotes the usual scalar product on $\mathbb{R}^n$ and $|x| = \sqrt{xx}$.

3. The Mellin transform $\gamma(s) = \int_0^\infty F(t^{1/n})\, t^s\, \frac{dt}{t}$, can be continued to a holomorphic function in a vertical strip which contains a real number $c > 1$ such that
$$\gamma(c) = \max_{\mathrm{Re}(s)=c} |\gamma(s)|.$$

Note that the Fourier transform of $f(|x|)$ is radially symmetric, which justifies to write it in the form $F(|y|)$ with a suitable function $F(t)$ of one real variable $t \geq 0$. The function F is known in the literature as Hankel transform of f. Note furthermore that the integral defining $\gamma(s)$ is absolutely convergent for $0 < \mathrm{Re}(s) < 1 + \frac{\varepsilon}{n}$. Finally note that any measurable function f satisfying 2. is automatically continuous.

Theorem 1. *Under the above hypothesis 1.–3. one has for λ_n, the maximal lattice kissing number of dimension n, the upper bound*

$$\lambda_n \leq \frac{2F(0)}{c(2c-1)\,\gamma(c)}\, \exp\big(1 + \frac{2c}{c-1} + c\,\frac{\gamma'}{\gamma}(c)\big)$$

Supplement to Theorem 1. *If, in addition, F is nonnegative then, for all $c > 1$ where the Mellin transform of F converges, one has*

$$\lambda_n \leq \frac{F(0)}{c(c-1)\gamma(c)}\, \exp\big(1 + \frac{c}{c-1} + c\,\frac{\gamma'}{\gamma}(c)\big).$$

Note that, for nonnegative F, the condition 3. is automatically satisfied with every c as in the supplement.

The first upper bound is worse than the second one by a factor strictly larger than $\mathrm{e} = \exp(1)$ and tending to e if c tends to infinity. However, the first bound applies to a larger variety of functions f than the second one since in Theorem 1 the Hankel transform F is not required to be nonnegative.

The simplest function satisfying the hypothesis 1.–3. is certainly $f(t) = \exp(-t^2)$. Here the supplement gives the bound $1.64^{n(1+o(1))}$, which is far off the Kabatiansky-Levenshtein bound. One can do better by using instead $f(t) = \max(0, 1-t^2)^p$ for $p \gg 0$. In this case, however, we cannot apply the bound of the supplement since the corresponding functions F are not nonnegative; but condition 3. is still satisfied (cf. section 3). By applying Theorem 1 to these functions one obtains

Theorem 2. $\lambda_n \leq 1.3592^{n(1+o(1))}$.

This is still worse than the Kabatiansky-Levenshtein bound. However, the methods used in this article are quite simple compared to the more subtle arguments in [K-L]. Moreover, we do not know whether it is not possible to make still a better choice for f for improving the asymptotic estimate. Whereas the asymptotic bound of Theorem 2 is not too bad, the bounds for specific dimensions using the functions $f(t) = \max(0, 1-t^2)^p$ are quite poor: $\lambda_2 \leq 12$, $\lambda_3 \leq 31$, etc.. Again it is possible that more suitable choices of f for particular dimensions may yield better results.

In the next section we shall prove Theorem 1 and its supplement, and in section 3 we shall give the details for deducing Theorem 2 from Theorem 1. Parts of this work use ideas already presented in [F-S] and in [S].

2 Proof of Theorem 1

Proof. For bounding λ_n to above we may and will restrict to lattices L in $\mathbb{R}^n$ with minimal length equal to 1. Thus we have to show that, for any such L, the number $a_1(L)$ of points in L with distance 1 to the origin is bounded above by the right hand side of the inequality of Theorem 1. We shall actually show that the Hurwitz zeta function

$$D(s) = \sum_{x \in L \setminus \{0\}} |x|^{-ns}$$

at $s = c$ (where the sum defining $D(s)$ converges absolutely since $c > 1$) is bounded to above by the right hand side. Since trivially

$$a_1(L) \leq D(c)$$

the estimate for $D(c)$ then proves the theorem.

Under the hypothesis 1 and 2 the Poisson summation formula is valid, i.e.

$$t \sum_{x \in L} F(t^{1/n}|x|) = \mathrm{g}(L)^{1/2} \sum_{x \in L^*} f(t^{-1/n}|x|),$$

where both sums are absolutely convergent [S-W], p. 252 (VII:Corollary 2.6). Here L^* denote the dual lattice of L (i.e. the set of all $y \in \mathbb{R}^n$ such that $yx \in \mathbb{Z}$ for all $x \in L$), and $\mathrm{g}(L)$ the Gram matrix of any $\mathbb{Z}$-basis of L.

Denote by $\theta(t)$ the sum on the left with the term $F(0)$ omitted. Thus the left hand side equals $t(F(0) + \theta(t))$. This is an increasing real valued function of t, since $f(t^{-1})$ (and hence the right hand side) is real valued and increasing by assumption 1.

For the Mellin transform $\Lambda(s)$ of $\theta(t)$ we find

$$\Lambda(s) = \int_0^\infty \theta(t)\, t^s \,\frac{dt}{t} = \gamma(s) \sum_{x\in L\setminus\{0\}} |x|^{-ns} = \gamma(s)D(s).$$

The Mellin transform is absolutely convergent for

$$1 < \mathrm{Re}(s) < 1 + \frac{\varepsilon}{n},$$

and in this domain interchanging the integral and the sum over x is indeed justified, as is easily deduced from the fact that the Mellin transform of $F(t^{1/n})$ is absolutely convergent for $0 < \mathrm{Re}(s) < 1 + \varepsilon/n$ (by hypothesis 2.), and that the Hurwitz zeta function of L converges for $\mathrm{Re}(s) > 1$.

Before giving the rest of the proof, we show how to deduce our estimate using a short argument which, however, is based on assumptions not holding true in general, but has the advantage of being rapid and suggestive. Assume that $\Lambda(s)$ can be analytically continued to a strip containing 0 and c, with poles only at 0 and 1, which are simple, and with residue $-F(0)$ at $s = 0$. This is a standard situation, which holds true for instance with $f(t) = \exp(-t^2)$. Then we may consider the function $g(s) = s(s-1)\Lambda(s)$. Note that $g(0) = F(0)$. Assume further that $\log g(s)$ is defined in $[0, c]$ and convex on this interval. Then

$$\log g(c) - \log g(0) \le c\,(\log g)'(c) = c\,\frac{g'}{g}(c).$$

Taking exponentials in this inequality gives

$$D(c) \le \frac{F(0)}{c(c-1)\gamma(c)} \exp\big(1 + \frac{c}{c-1} + c\frac{\gamma'}{\gamma}(c) + c\frac{D'}{D}(c)\big).$$

Dropping the term containing D' (which is negative since $D(s)$ is decreasing for $s > 1$) we recognise the estimate for $D(c)$ as in the supplement of Theorem 1.

The mentioned assumption about $\Lambda(s)$ will in general be false. Nevertheless, if F is nonnegative, the inequality for $g(s)$ holds still true for every $c > 1$ where the Mellin transform of F converges. For the proof we rewrite it as

$$F(0) \ge g(c) \exp\big(-c\frac{g'}{g}(c)\big),$$

This inequality is equivalent to the statement that for all $a > 0$ one has

$$F(0) \ge \big(g(c) - cg'(c) + cg(c)\log a\big)a^{-c} = -c^2\,\frac{d}{ds}\left[\frac{a^{-s}}{s} g(s)\right]_{s=c}.$$

Indeed, taking $\log a = \frac{g'}{g}(c)$ gives the first inequality, and for this a the right hand side of the second one attains its maximum. For proving the latter estimate we consider, for fixed $s > 1$ (where $\gamma(s)$ converges), the function

$$I(a) = aF(0)/s^2 - a \int_0^a \theta(t) \log(t/a)(t/a)^s \, \frac{dt}{t} = \int_0^\infty (F(0) + \theta(at))at \, w(t) \, \frac{dt}{t},$$

where $w(t) = -t^{s-1} \log \max(1, t)$. The second identity is easily justified by the substitution $t \leftarrow at$. Since $w(t)$ is nonnegative and $F(0) + \theta(t)t$ is increasing, the function $I(a)$ is an increasing function. Hence $I'(a) \geq 0$. Using the first formula for $I(a)$ for computing the derivative we thus obtain

$$F(0)/s^2 \geq -\int_0^a \theta(t) \; (1 + (s-1)\log(t/a)) \, (t/a)^s \, \frac{dt}{t} = -\frac{d}{ds} \frac{a^{-s}}{s} \int_0^a \theta(t) t^s \, \frac{dt}{t}.$$

Since $F(t)$ and hence $\theta(t)$ is nonnegative this inequality remains valid if we replace $\int_0^a$ by $\int_0^\infty$. Writing c for s this is then the desired inequality, which proves the supplement to Theorem 1.

In the general case, i.e. for not necessarily nonnegative F, we shall be able to prove an estimate similar to the one above, however with a factor $1/2$ and a slightly different g. Namely, we shall prove

$$F(0) \geq \frac{1}{2} \, g(c) \exp\big(-c\frac{g'}{g}(c)\big),$$

where

$$g(s) = \frac{2c-1}{2c-1-s} \, s(s-1)\Lambda(s).$$

From this we deduce as before an upper bound for $D(c)$ and then Theorem 1.

As before the inequality for $F(0)$ is equivalent to the statement that for all $a > 0$ one has

$$F(0) \geq -\frac{c^2}{2} \frac{d}{ds} \big[\frac{a^{-s}}{s} g(s)\big]_{s=c}.$$

For proving this inequality we set

$$\begin{aligned} H(a) = aF(0)/c^2 + \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c-\varepsilon} \frac{a^{1-s} g(s)}{s(s-1)} \cdot \frac{ds}{(s-c)^2} \\ = \int_0^\infty (F(0) + \theta(at))at \, v(t) \, \frac{dt}{t}, \end{aligned}$$

where

$$v(t) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=c-\varepsilon} \frac{2c-1}{2c-1-s} \cdot \frac{t^{s-1} \, ds}{(s-c)^2},$$

The second identity is easily justified by making on the right hand side the substitution $t \leftarrow t/a$, replacing $v(t)$ by its integral representation, interchanging integrals and using finally $\int_0^\infty v(t)\,dt = 1/c^2$. For ε we may choose and positive real number such that $c-\varepsilon > 1$ and such that the Mellin transform of $F(t)$ (and hence of $\theta(t)$) converges absolutely at $s = c-\varepsilon$.

By a simple calculation we see

$$v(t) = \frac{(2c-1)}{(c-1)^2} \cdot \begin{cases} t^{c-1}\big(-\log t^{c-1} + t^{c-1} - 1\big) & \text{if } t < 1 \\ 0 & \text{if } t \geq 1 \end{cases}.$$

Hence $v(t)$ is nonnegative, and since $(F(0)+\theta(t))t$ is increasing, we see that $H(a)$ is an increasing function. Hence $H'(a) \geq 0$, i.e.

$$F(0)/c^2 \geq \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c-\varepsilon} \frac{a^{-s}g(s)}{s}\cdot\frac{ds}{(s-c)^2}.$$

Now, for $s = c+it$ with real t, one has

$$\mathrm{Re}\,\frac{a^{-s}g(s)}{s} \leq \Big|\frac{a^{-s}g(s)}{s}\Big| \leq \frac{g(c)\,a^{-c}}{c}$$

as follows easily from the assumption that $|\gamma(s)| \leq \gamma(c)$ for $s = c+it$. We may hence apply the following lemma to estimate the last integral to below by $-\frac{1}{2}\frac{d}{ds}[\frac{g(s)}{s}a^{-s}]_{s=c}$, which is the desired inequality. This proves the theorem. $\square$

The following lemma was proven, in a slightly different form, in [F-S].

Lemma. *Let $f(s)$ be a bounded and holomorphic function in some strip $a < \mathrm{Re}\,(s) < b$, real valued for real s. Assume that for some $a < c < b$ we have*

$$\sup_{t\in\mathbb{R}} \mathrm{Re} f(c+it) = f(c).$$

Then, for all $\varepsilon > 0$ with $a < c-\varepsilon$, one has

$$\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=c-\varepsilon} \frac{f(s)\,ds}{(s-c)^2} \geq -\frac{1}{2}f'(c).$$

Proof. Since $f(s)$ is bounded, the integrand of the integral in question is $O(t^{-2})$ for $t = \mathrm{Re}\,s \to \pm\infty$. Hence we can replace the path of integration by the line segment from $c-i\infty$ to $c-i\delta$, then along the left half circle with centre c to $c+i\delta$, and finally the line up to $c+i\infty$. Here δ is any positive number. For the integral along the half circle γ we find

$$\frac{1}{2\pi i}\int_\gamma = \frac{1}{\delta}\int_{3/4}^{1/4} f\big(c+\delta e^{2\pi i t}\big)\,e^{-2\pi i t}\,dt = \frac{1}{\pi\delta}f(c) - \frac{1}{2}f'(c) + O(\delta).$$

For the integrals along the line segments we find, using $\overline{f(z)} = f(\overline{z})$ (since this is true for real s by assumption),

$$\frac{1}{2\pi i}\Big(\int_{c-i\infty}^{c-i\delta} + \int_{c+i\delta}^{c+i\infty}\Big) = -\frac{1}{\pi}\int_{\delta}^{\infty} \frac{\mathrm{Re} f(c+it)\,dt}{t^2} \geq -\frac{1}{\pi\delta} f(c).$$

For the inequality we used $\mathrm{Re} f(c+it) \leq f(c)$. On taking $\delta \to 0$ the lemma follows. $\square$

3 Proof of Theorem 2

Proof. For $p \geq 0$, the function

$$f(t) := \max(0, 1-t^2)^p$$

is nonnegative, decreasing and continuous. We show that f satisfies the assumptions of Theorem 1.

One has

$$F(t) = C\,\Gamma\big(q\big)\, J_{q-1}(2\pi t)/(\pi t)^{q-1} \qquad (\ q = \frac{n}{2} + p + 1\ )$$

with J_{q-1} being the Bessel function of order $q-1$ [S-W], p. 171 (IV:Theorem 4.15), and with a positive constant C. Note that $F(0) = C$. Using

$$J_{q-1}(t) = O(t^{-1/2})$$

[S-W], p. 158 (IV:Lemma 3.11), we see that F satisfies the hypothesis 2. of Theorem 1 as long as

$$q > n + \frac{1}{2} \qquad (\text{i.e. } p > \frac{n-1}{2}).$$

In the following we assume this inequality. The Mellin transform of $F(t^{1/n})$ equals

$$\gamma(s) = C\,\frac{n}{2}\pi^{-ns}\,\frac{\Gamma\big(q\big)\Gamma\big(\frac{ns}{2}\big)}{\Gamma\big(q - \frac{ns}{2}\big)}$$

([S-W], p. 174 or Titchmarch) The hypothesis 3. is therefore satisfied for $c = q/n$ since, for any s with real part c, one has $|\gamma(s)| = \gamma(c)$.

Thus, by Theorem 1, we obtain the estimate

$$\lambda_n \leq \frac{4n}{q(2q-n)\,\Gamma(q)} \exp\big(1 + \frac{2q}{q-n} + q\,\psi(\frac{q}{2})\big).$$

Using

$$\log \Gamma(q) = q \log q - q + o(q), \quad \psi(q) = \log q + O(q^{-1}) \qquad (q \to \infty),$$

we obtain, for all fixed small $10^{-4} \geq \varepsilon > 0$, by setting $q = (1+\varepsilon)n$, the estimate

$$\lambda_n \leq \exp\big(n(1+\varepsilon)(1-\log 2) + o(n)\big) \leq 1.3592^{n(1+o(1))}$$

as $n \to \infty$. This proves Theorem 2. $\square$

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